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SCIENTIFIC COMMUNICATIONS

Inversion of a Discontinuous Travel–Time Curve of Refracted Wave

V. Yu. Burmin

Schmidt Joint Institute of Physics of the Earth, Russian Academy of Sciences, Bol'shaya Gruzinskaya ul. 10, Moscow, 123810 Russia

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In 1932, in his study of travel–time curves in an elastic medium containing a waveguide [1], Slichter showed that this curve remains unchanged, as the layers in the waveguide are arbitrarily rearranged. Thus, it was proven that the solution of the kinematic seismic inversion problem is nonunique.

In 1967, Gerver and Markushevich [2] derived a formula for inversion of the travel–time curve, which allowed for a velocity distribution in a waveguide, but was useless for finding the distribution itself. Based on this formula, they proved that the velocity is determined nonuniquely both in the waveguide and below it and that neither the minimum velocity nor the maximum thickness of the waveguide can be found. Note that the problem statement included seismic rays corresponding only to refracted and simple reflected waves and did not consider grazing rays that produce head waves. However, the solution of the problem was sought in the class of twice piecewise smooth velocity functions and so the presence of grazing rays was assumed. In other words, a set of velocity functions was considered that correspond to travel–time curves of refracted, head, and simple reflected seismic waves, but considered the travel–time curves only if refracted and supercritical reflected waves were inverted. Then, even with waveguides absent, the one-to-one correspondence between sets of travel–time curves and velocity distributions does not obviously take place, as proven by Gerver and Markushevich [2] for the case of ray focusing.

Based on the analytical continuation of travel–time curves, Geiko [3] studied nonuniqueness of seismic kinematic inversion. He proved the uniqueness theorems for the determination of the distribution function $\sigma(u)$ in a wide class of velocity functions. In other words, the set of solutions was shown to be of the Slichter type; i.e., the velocity function outside the waveguide is determined uniquely, and inside the waveguide is determined accurate to a rearrangement of elementary waveguide layers. Furthermore, both the minimum velocity in and thickness of the waveguide are determined uniquely.

This paper derives equations for determination of the velocity function from a discontinuous travel–time curve, which are different from those obtained by Geiko [3], and presents a numerical solution of the equations.

Travel–time curve for a seismic wave propagating in a laterally homogeneous (vertically nonhomogeneous) medium is described by the equations

$$x(p) = 2p\int_0^{z(p)} \frac{dz}{\sqrt{u^2(z)-p^2}},\qquad t(p) = 2\int_0^{z(p)} \frac{u^2(z)dz}{\sqrt{u^2(z)-p^2}}, \tag{1}$$

where $u(z) = v^{-1}(z)$ is the reciprocal of the velocity of seismic wave ($u(z)$ can be interpreted as an index of refraction relative to unit velocity), p is the ray parameter ($p = t'$), and $z(p)$ is the depth of the deepest penetration of the ray with the parameter p.

We consider only refracted waves generated by a surface source, so that different maximum depths $z(p)$ of penetration of rays correspond to different values of the ray parameter p. In the general case, the curve $t(x)$ has discontinuities that can be attributed to the presence of waveguides, velocity jumps, and layers with a constant velocity [4]. Assume that the first branch of the travel–time curve begins at the source (generally speaking, this condition is not necessary). Then this branch can be used to retrieve the velocity distribution up to a depth z^* with the use of the known Herglotz–Wiechert–Chibisov formula [5]. Taking into account that $\bar{z}^*$ is a maximum penetration depth of the ray corresponding to the second branch of the travel–time curve and reducing the remaining branches to the depth $\bar{z}^*$, obtain

$$x(p, z^*) = x(p) - 2p\int_0^{z^*} \frac{dz}{\sqrt{u^2(z)-p^2}}$$

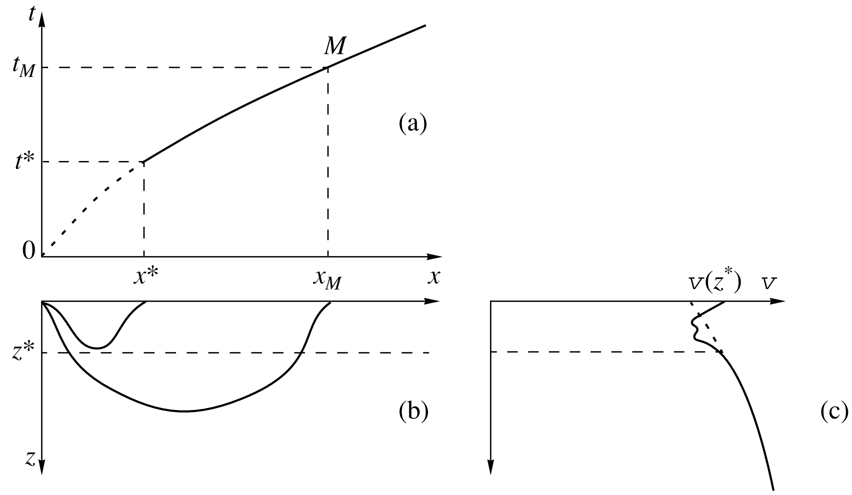

Fig. 1. (a) Travel–time curve for refracted waves. (b) Seismic rays in an elastic medium with waveguide. (c) Seismic rays in an elastic medium with an arbitrary velocity distribution.

$$= 2p\left\{\int\limits_{z^*}^{\bar{z}^*} + \int\limits_{\bar{z}^*}^{z(p)}\right\}\frac{dz}{\sqrt{u^2(z)-p^2}}, \tag{1'}$$

$$t(p, z^*) = t(p) - 2\int\limits_0^{z^*}\frac{u^2(z)dz}{\sqrt{u^2(z)-p^2}}$$

$$= 2\left\{\int\limits_{z^*}^{\bar{z}^*} + \int\limits_{\bar{z}^*}^{z(p)}\right\}\frac{u^2(z)dz}{\sqrt{u^2(z)-p^2}}.$$

Consider the second branch of the travel–time curve $t(x)$ shown (after reducing) in Fig. 1. If the break in this curve is due to the positive velocity jump, the point $(x(q_0), t(q_0))$ corresponding to the beginning of the second travel–time branch is coincident with the origin of coordinates. However, if the break in the travel–time curve is caused by a waveguide or a layer with constant velocity (not of waveguide type), then $x(q_0) > 0$ and $t(q_0) > 0$. In the case of a layer of constant velocity, the layer parameters v^* and Δz are determined by the formulas [4]

$$v^* = \sqrt{\bar{v}v_K}, \quad \Delta z = \frac{x(q_0)}{2}\sqrt{\frac{v_K}{\bar{v}} - 1},$$

where

$$v_K^{-1} = q_0 = t'[x(q_0)]; \quad \bar{v} = \frac{x(q_0)}{\bar{t}(q_0)}.$$

In the case of a waveguide, as mentioned above, an infinite set of velocity functions satisfying an observed travel–time curve (with break) exists. The velocity inside the waveguide is an arbitrary twice piecewise smooth function, and the velocity outside the waveguide is a strictly monotonic, increasing, twice piecewise smooth function. Let us consider the function $H(u) = \mathrm{mes}\{z\colon z \in [z^*, \bar{z}^*], v^{-1}(z) \ge u\}$. By definition, the measure $H(u)$ does not decrease, vanishes at $-\infty < u \le \bar{u}^*$, and equals $h = \bar{z}^* - z^*$ at $u^* \le u < \infty$, where $\bar{u}^*$ and u^* are the respective minimum and maximum values of the index of refraction in the waveguide.

Since the function $u = u(z)$ strictly decreases outside the waveguide, an inverse function $z = z(u)$ exists. Then the travel–time equations are written as a sum of Rieman–Stieltjes integrals,

$$x(p, z^*) = 2p\left\{\int\limits_p^{q_0}\frac{\varphi(u)du}{\sqrt{u^2-p^2}} + \int\limits_{\bar{u}^*}^{u^*}\frac{dH(u)}{\sqrt{u^2-p^2}}\right\},$$

$$t(p, z^*) = 2\left\{\int\limits_p^{q_0}\frac{\varphi(u)u^2du}{\sqrt{u^2-p^2}} + \int\limits_{\bar{u}^*}^{u^*}\frac{u^2dH(u)}{\sqrt{u^2-p^2}}\right\}, \tag{2}$$

where $\varphi(u) = -dz/du \ge 0$; $dH(u) \ge 0$.

Multiplying the right- and left-hand sides of the first equation in (2) by $1/\sqrt{w^2-p^2}$ and integrating from q to q_0, where q and q_0 are the ray parameters corresponding to the end and beginning of the given branch of the travel–time curve (Fig. 1), we find

$$\int\limits_q^{q_0}\frac{x(p, z^*)dp}{\sqrt{w^2-p^2}}$$

$$= \int\limits_q^{q_0}\frac{2p}{\sqrt{w^2-p^2}}\left\{\int\limits_p^{q_0}\frac{\varphi(u)du}{\sqrt{u^2-p^2}} + \int\limits_{\bar{u}^*}^{u^*}\frac{dH(u)}{\sqrt{u^2-p^2}}\right\}dp,$$

where $q \le p \le q_0 \le \bar{u}^* \le w \le u^*$.

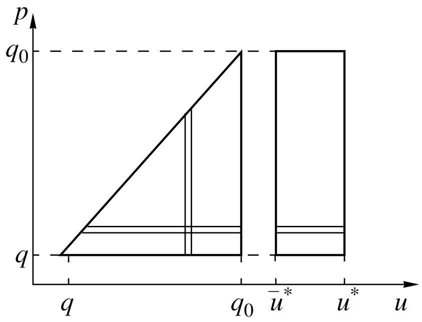

Fig. 2. Domain of integration in the inversion of a discontinuous travel–time curve for refracted waves.

Interchanging the inner and outer integrals by the Dirichlet rule (domain of integration is shown in Fig. 2) gives

$$\int_q^{q_0} \frac{x(p, z^*)dp}{\sqrt{w^2 - p^2}} = \int_q^{q_0} \frac{2p}{\sqrt{w^2 - p^2}}$$

$$\times \left\{ \int_p^{q_0} \frac{\varphi(u)du}{\sqrt{u^2 - p^2}} + \int_{\bar{u}^*}^{u^*} \frac{dH(u)}{\sqrt{u^2 - p^2}} \right\} dp$$

$$= \int_q^{q_0} \varphi(u) \int_q^u \frac{2p\,dp}{\sqrt{(u^2 - p^2)(w^2 - p^2)}} du \tag{3}$$

$$+ \int_{\bar{u}^*}^{u^*} dH(u) \int_q^{q_0} \frac{2p\,dp}{\sqrt{(u^2 - p^2)(w^2 - p^2)}}$$

$$= \int_q^{q_0} \varphi(u) K_1(u, w) du + \int_{\bar{u}^*}^{u^*} K_2(u, w) dH(u),$$

where

$$K_1(u, w) = 2\ln\frac{\sqrt{w^2 - q^2} + \sqrt{u^2 - q^2}}{\sqrt{w^2 - u^2}},$$

$$q \le u \le q_0 \le \bar{u}^* \le w \le u^*;$$

$$K_2(u, w) = 2\ln\frac{\sqrt{w^2 - q^2} + \sqrt{u^2 - q^2}}{\sqrt{w^2 - q_0^2} + \sqrt{u^2 - q_0^2}},$$

$$q \le q_0 \le \bar{u}^* \le u, w \le u^*.$$

Multiplying both sides of the second equation in (2) by $p/\sqrt{w^2 - p^2}$ and integrating again from q to q_0, we obtain

$$\int_q^{q_0} \frac{t(p, z^*)dp}{\sqrt{w^2 - p^2}} = \int_q^{q_0} \varphi(u) u^2 K_1(u, w) du$$
$$+ \int_{\bar{u}^*}^{u^*} u^2 K_2(u, w) dH(u). \tag{4}$$

Thus, we found a system of two integral equations of the Fredholm type for the two functions $\varphi(u)$ and $H(u)$, which must satisfy the conditions

$$\varphi(u) \ge 0, \quad dH(u) \ge 0. \tag{5}$$

We consider the numerical solution of equations (3) and (4). The segment $[q, q_0]$ is divided into $N - 1$ subsegments by N points, and the function $\varphi(u)$ is set equal to a positive constant φ_j ($j = 1, 2, \ldots, N - 1$) at each of the subsegments. The segment $[\bar{u}^*, u^*]$ is also divided into $N - 1$ subsegments by N points, and the function of jumps ΔH_j ($j = 1, 2, \ldots, N - 1$) is introduced on the segment $[\bar{u}^*, u^*]$. Then (3) and (4) are written as a system of $2N$ equations, linear for the $2N - 2$ unknowns φ_j and ΔH_j and nonlinear for the unknowns $w_1 = \bar{u}^*$ and $w_N = u^*$. The system can be written in the form

$$Ay - f = 0, \tag{6}$$

where $f^T = \{f_i\}$ ($i = 1, 2, \ldots, N, N + 1, \ldots, 2N$), $y^T = \{\varphi_j\}$ ($\varphi_j \ge 0$), ($j = 1, 2, \ldots, N - 1$) and $y^T = \{\Delta H_j\}$ ($\Delta H_j \ge 0$), ($j = N, N + 1, \ldots, 2N - 2$) are vectors, and $A = \{a_{ij}\}$ is a $2N$ by $2(N - 1)$ matrix. Furthermore,

$$f_i = \int_q^{q_0} \frac{x(p, z^*)dp}{\sqrt{w_i^2 - p^2}} \quad \text{for} \quad i = 1, 2, \ldots, N,$$

$$f_i = \int_q^{q_0} \frac{t(p, z^*)dp}{\sqrt{w_{i-N}^2 - p^2}} \quad \text{for} \quad i = N + 1, N + 2, \ldots, 2N,$$

$$a_{ij} = 2\int_{q_j}^{q_{j+1}} \ln\frac{\sqrt{w_i^2 - q^2} + \sqrt{u^2 - q^2}}{\sqrt{w_i^2 - u^2}} du$$

$$\text{for} \begin{cases} i = 1, 2, \ldots, N \\ j = 1, 2, \ldots, N - 1, \end{cases}$$

$$a_{ij} = 2\int_{w_{j-N+1}}^{w_{j-N+2}} \ln\frac{\sqrt{w_i^2 - q^2} + \sqrt{u^2 - q^2}}{\sqrt{w_i^2 - q_0^2} + \sqrt{u^2 - q_0^2}} du$$

$$\text{for} \begin{cases} i = 1, 2, \ldots, N \\ j = N, N + 1, \ldots, 2N - 2, \end{cases}$$

$$a_{ij} = 2\int_{q_j}^{q_{j+1}} u^2 \ln\frac{\sqrt{w_{i-N}^2 - q^2} + \sqrt{u^2 - q^2}}{\sqrt{w_{i-N}^2 - u^2}} du$$

$$\text{for } \begin{cases} i = N+1, N+2, \dots, 2N \\ j = 1, 2, \dots, N-1, \end{cases}$$

$$a_{ij} = 2\int_{w_{j-N+1}}^{w_{j-N+2}} u^2 \ln\frac{\sqrt{w_{i-N}^2 - q^2} + \sqrt{u^2 - q^2}}{\sqrt{w_{i-N}^2 - q_0^2} + \sqrt{u^2 - q_0^2}} du$$

$$\text{for } \begin{cases} i = N+1, N+2, \dots, 2N \\ j = N, N+1, \dots, 2N-2, \end{cases}$$

$$w_i = w_1 + \frac{w_N - w_1}{N-1}(i-1) = w_1 + \frac{\Delta w}{N-1}(i-1),$$

$$w_1 \ge u(z^* - 0), \quad \Delta w \ge 0.$$

System (6) is solved as follows. Isolating the unknowns y_j in the first $2N - 2$ equations, for example by the Gaussian elimination, and substituting them in the remaining two equations, we find the system of two equations for w_1 and Δw

$$F(w_1, \Delta w) = 0, \quad G(w_1, \Delta w) = 0, \quad w_1 \ge u(z^* - 0), \quad \Delta w \ge 0. \tag{7}$$

As mentioned above, Geiko [3] showed that the minimum velocity in the waveguide and its thickness are uniquely determined for a wide class of velocity functions. In particular, the uniqueness takes place for piecewise analytical functions, and therefore, the problem on determination of w_1 and Δw has a unique solution. Solving system (7), we get the values of w_1 and Δw and so w_i ($i = 1, 2, \dots, N$). Substituting these w_i in the first $2N - 2$ equations of (6), we determine y_j ($j = 1, 2, \dots, 2N - 2$).

The function $H(u)$ on the segment $[\bar{u}^*, u^*]$ is determined from the relations $H(u) = H_{j-N} + \Delta H_{j-N}$, $u \in [u_j, u_{j+1}]$, $\Delta H_{j-N} \ge 0$, $j = N+1, N+2, \dots, 2N-1$, $H_1 = H(\bar{u}^*) = 0$. Considering that $H(u^*) = h$ and

$$\varphi_j = -\frac{z_{j+1} - z_j}{u_{j+1} - u_j}, \quad z_N = h,$$

we find the function $z(u)$ on the segment $[q, q_0]$ from the relations

$$z(u) = z_{j+1} + \varphi_j(u_{j+1} - u), \quad u \in [u_j, u_{j+1}],$$

$$\varphi_j \ge 0, \quad j = N-1, \dots, 1.$$

In (3) and (4), the upper and lower limits of integration of the second terms on the right-hand sides equal the maximum and minimum values, respectively, of the index of refraction in the waveguide, which were determined from the set of nonlinear equations (7). By definition, we have $H(u) = 0$ at $-\infty < u \le \bar{u}^*$ and $u^* < u \le \infty$. Therefore, if the segment $[\bar{u}^*, u^*]$ is extended up to the segment $[\tilde{\bar{u}}^*, \tilde{u}^*]$, where $q_0 \le \tilde{\bar{u}}^* = \tilde{u}(\bar{z}^* - 0) \le \bar{u}^*$ and $\tilde{u}^* = \tilde{u}(z^* + 0) \ge u^*$, the values of the integrals in (2) do not change, and, because of the uniqueness of the solution, the obtained distribution $\tilde{H}(u)$ differs from the actual one only in layers of zero thickness. Thus, the values u^* and $\bar{u}^*$ may arbitrarily be specified in the intervals $[u(z^* + 0), u_{\max}]$ and $[u(z^* + 0), u(\bar{z}^* - 0)]$, where $u_{\max}$ is an arbitrary value satisfying the inequality $u(z^*) \le u_{\max}$. Substitution of the corresponding w_i into (6) ($w_1 = \tilde{u}^*$, $w_N = \tilde{\bar{u}}^*$) gives the system of $2N$ linear algebraic equations for $2N - 2$ unknowns y_j ($j = 1, 2, \dots, 2N - 2$),

$$Ay = f. \tag{8}$$

Taking a vector y, providing the minimum of the functional

$$J = \|Ay - f\|^2, \tag{9}$$

as a solution of system (8) and using the condition of nonnegativity of the unknown parameters,

$$y_j \ge 0, \tag{10}$$

we arrive at a problem of quadratic programming, namely, that of minimization of quadratic functional (9) at linear constraints (10). The problem is known to have a unique solution, since the functional J convex in y is bounded from below and continuous on the convex set $Y = \{y_j | y_j \ge 0\}$.

Solving the problem of (9) and (10), for example, by the conjugate gradient method [6], we find y_j.

Since we solve an approximate finite-dimensional problem instead of determining $\varphi(u)$ and $H(u)$ in the infinite-dimensional space, layers of nonzero thickness, with index of refraction ranging within $u^* \le u \le u_{\max}$ and $u(z^* - 0) \le u \le \bar{u}^*$, inevitably appear at arbitrary values of $\tilde{\bar{u}}^*$, and $\tilde{u}^*$ and arbitrary partition of the segment $[\tilde{\bar{u}}^*, \tilde{u}^*]$. To improve the values of $\bar{u}^*$ and u^*, we use the following iteration procedure. First, the problem is solved for some $\bar{u}_0^*$ and u_0^*. Then the extreme values u_j, for which y_j are zero, are accepted as $\bar{u}_1^*$ and u_1^*. The latter are used in the next iteration, and so on, until the extreme values of y_1 and y_{N+M-2} are nonzero. Obviously, the denser the subdivision of the segment $[\bar{u}^*, u^*]$, the more efficient the procedure.

Thus, the function $u = u(z)$ was determined in the interval $[z^*, z(p)]$ and, in particular, in the waveguide, i.e., in the interval $[z^*, \bar{z}^*]$. The set of velocity functions in the waveguide, satisfying a given travel–time curve, is a set of equimeasurable functions in the interval $[z^*, \bar{z}^*]$, for which the maximum value of $u(z)$ in the waveguide and its thickness are the same. If the travel–time curve has n breaks, the function $u = u(z)$ is determined by the method described above in the whole domain penetrated by seismic rays. Of course, the aforesaid is applicable not only to breaks of travel–time curves due to waveguides, but also to fragments of the curves due to incomplete data. Therefore, the velocity curve is retrieved in a class of monotonic functions from any fragment of a travel–time curve. Obviously, the longer the fragment, the more stable the inversion.

Travel–time equation (1) involves the ray parameter p that is the gradient of the travel–time curve at the emergence point of a ray. To determine p, the travel–time curve given by a discrete set of points with errors should be differentiated, and this is generally an ill-posed problem. However, the travel–time curve for a seismic wave propagating in a vertically nonhomogeneous elastic medium is known to be a convex function. Therefore, to determine the gradient of the travel–time curve, the set of experimental points should be smoothed by a convex curve. For this we use a convex cubic spline $T(x)$ [7] to minimize the functional

$$S = ||BT'' - y||^2 \tag{11}$$

subject to the constraints

$$T_i'' \le 0, \tag{12}$$

where T_i'' are the second derivatives of the spline at points ($i = 1, 2, \ldots, n$); B is an n^* by n quadratic matrix whose elements depend only on x_i; y is vector of free terms depending on time values of the travel–time curve $t(x_i)$ at observation points x_i and on the boundary conditions for the smoothing spline. The problem of minimizing functional (11) under linear constraints (12) is also a problem of quadratic programming.

In conclusion, note that the solution described above is based on the assumption that the velocity function outside the waveguide is strictly monotonic. In reality, the depth dependence of seismic velocity can have zero gradient; i.e., the velocity function can be not strictly monotonic. In this case, grazing rays are known to generate head waves. If for solving the inverse kinematic seismic problem we use a wider class of velocity functions, in particular, the class of nonstrictly monotonic functions, travel–time data of a refracted wave should be interpreted on a wider basis, including also travel–time curves of head waves produced by grazing rays. Otherwise, the set of solutions would be much wider than Slichter's set, the result obtained by Gerver and Markushevich (e.g., [2]).

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