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A New Approach to Teleseismic Hypocenter Location

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The usual procedure in seismological practice for hypocenter location is to minimize a functional involving residuals between observed and theoretical travel times of seismic waves. This approach is essentially equivalent to the minimization of a functional involving residuals of the respective hypocentral distances, which can in turn be represented as radial vectors with origin at observation sites. The difference between the lengths of two radial vectors is independent of their directions; consequently, the determination of hypocentral parameters by minimizing the residuals of the lengths of these vectors may prove unstable. It is suggested that hypocenter location be based on the lengths of the vectorial differences between the radial vectors corresponding to observed and theoretical travel times rather than on the differences between the lengths of these vectors. This approach enhances stability in hypocenter location. This solution takes account of the Earth's ellipticity on the assumption that seismic velocity is an arbitrary function of the distance to the Earth's center.

Equations that Relate Hypocenter Coordinates and the Coordinates of Seismic stations on an Ellipsoid. The usual procedure in seismological practice for determining the hypocenter coordinates of earthquakes recorded by the world seismograph network is to minimize the following functional of residuals:

$$S_t = \sum_{i=1}^{n} (t_i - \tilde{t}_i)^2, \tag{1}$$

where t_i and $\tilde{t}_i$ are theoretical and observed travel times, t_i being a function of epicentral distance, depth of focus, and velocity structure along the paths.

If the travel time uncertainty δt is fixed, then all hypocenter coordinates with $S_t^{1/2} \leq \delta t$ will belong to some region ω which may generally be multiply connected. We now write down equations relating the coordinates of points in ω, that is, the coordinates of potential hypocenters and the coordinates of seismograph stations. Let n stations situated at the ground surface have geographic coordinates φ_i, λ_i and elevations Δh_i above sea level, and the hypocenter, coordinates φ_0, λ_0 and depth H_0. A set of cartesian coordinates is assumed to have the origin at the Earth's center. The $0Z$ axis is along the polar axis of the terrestrial ellipsoid, the $0X$ axis, along the intersection of the equatorial plane and the plane of zero longitude, and the $0Y$ axis lies in the equatorial plane but along the meridian

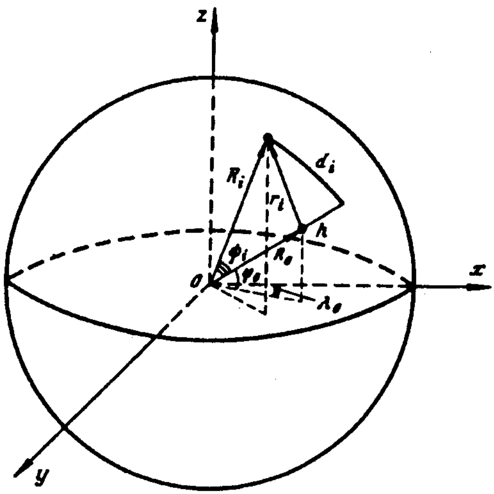

Figure 1 Transition from geographical to geocentric coordinates.

at right angles to the plane of zero longitude. We can write a set of nonlinear equations relating the coordinates of the hypocenter and the stations:

$$(X_0 - x_i) + (Y_0 - y_i) + (Z_0 - z_i)^2 = r_i^2. \quad (2)$$

Here X_0, Y_0, Z_0 are hypocenter coordinates; x_i, y_i, z_i are station coordinates; $\mathbf{r}i = \mathbf{R}_i - \mathbf{R}_0$;

$$|\mathbf{R}_i| = R_i = (x_i^2 + y_i^2 + z_i^2)^{1/2}, \quad |\mathbf{R}_0| + R_0 = (X_0^2 + Y_0^2 + Z_0^2)^{1/2};$$
$$(i=1, 2, \ldots, n).$$

The geographic and cartesian coordinates on the surface of the ellipsoid are related as follows [4]

$$x = \frac{a\cos\varphi\cos\lambda}{(1 - e^2\sin\varphi)^{1/2}}, \quad y = \frac{a\cos\varphi\sin\lambda}{(1 - e^2\sin\varphi)^{1/2}}, \quad z = \frac{a(1 - e^2)\sin\varphi}{(1 - e^2\sin\varphi)^{1/2}}, \quad (3)$$

where a is the semi-major axis; $e^2 = (a^2 - b^2)/a^2$ is the first eccentricity of the meridional ellipse; b is the semi-minor axis of the ellipsoid.

The solution of (2) is sought on the surface of a sphere of radius R_0. The following relations hold:

$$X_0 = R_0\cos\varphi\cos\lambda, \quad Y_0 = R_0\cos\varphi\sin\lambda, \quad Z_0 = R_0\sin\varphi. \quad (3a)$$

We write out (2) and use the relations for R_i and R_0 (Figure 1). The result is

$$X_0 x_i + Y_0 y_i + Z_0 z_i = 0.5(R_0^2 + R_i^2 - r_i^2). \tag{4}$$

The right-hand side of (4) includes

$$r_i^2 = R_0^2 + R_i^2 - 2R_0 R_i \cos\psi_i,$$

where $\psi_i = d_i/R_i$ is the angle between $\mathbf{R}_0$ and $\mathbf{R}_i$, d_i being epicentral distances.

Denote $r_0 = R_0$. Substituting ψ_i into the expressions for r_i^2 and r_i^2 in (4) and dividing both sides by $r_0 R_i$, we get

$$Uu_i + Vv_i + Ww_i = \cos(d_i/R_i) = \cos(\psi_i), \tag{5}$$

where $U = X_0/\mathbf{r}_0$; $V = Y_0/r_0$; $W = Z_0/r_0$; $w_i = x_i/R_i$; $v_i = y_i/R_i$; $w_i = z_i/R_i$; $r_0 = R_3 - h$; h is a hypocentral depth measured from the ground surface; $R_3 = (X^2 + Y^2 + Z^2)^{1/2}$ is the Earth's radius at the point of interest on the ground surface; X, Y, Z are given by (3). The hypocentral coordinates must remain within the ellipsoid, that is, the sum of squares of the unknowns U, V, W must satisfy the inequality

$$U^2 + V^2 + W^2 \leq [1 + e^2 \sin^2\varphi - 2e^2(\sin\varphi)/a]/(1 - e^2 \sin\varphi). \tag{6}$$

Equations (5) are linear algebraic equations in three unknowns U, V, W. The right-hand side of (5) contains $d_i = R_i\psi_i$, which are epicentral distances on the sphere of radius R_i; these quantities are functions of h and are to be calculated beforehand. Solving (5), we get

$$X_0 = U/r_0; \quad Y_0 = V/r_0; \quad Z_0 = W/r_0;$$
$$R_0 = (X_0^2 + Y_0^2 + Z_0^2)^{1/2}; \quad H = R_3 - R_0.$$

Obviously, every point in ω defines a set of h and d_i. Accordingly, for every set of h and d_i there is a corresponding solution $\{U, V, W\}$ of (5) whose totality generates a set Ω with elements H and $D_i = R_i\Psi_i$ $(i = 1, n)$, where Ψ_i is the angular distance between the earthquake epicenter and the i-th station.

It is reasonable to take for the solution those values of d_i, hence of h, which minimize the distance between ω and Ω in a given metric on the one hand and, on the other, satisfy the requirement $S_t^{1/2} \leq \delta t$.

We define the distance between two points M_1 and M_2 within the sphere as

$$\rho = (R_{sph}^2(\psi_1 - \psi_2)^2 + (r_1 - r_2)^2)^{1/2},$$

where R_{sph} is the radius of the sphere; r_1 and r_2 are distances along the radius from the center of the sphere to M_1 and M_2; $\psi_1 - \psi_2$ is the shortest angular distance between points M_1 and M_2. Then the sum of squares of the distances between points in ω and Ω can be written as

$$S = \sum_{i=1}^{n} [(D_i - d_i)^2 + (H - h)^2] = \sum_{i=1}^{n} [R_i^2(\Psi_i - \psi_i) + (R_0 - r_0)^2]. \tag{7}$$

Thus, the problem of hypocenter location reduces to the determination of region ω and the associated Ω, and to the minimization of (7).

The problem can be solved simply for a spherically symmetric distribution of seismic velocity in the Earth. The velocity is then a function of the distance from the Earth's center to the point of interest only, and the ψ_i are given by the explicit formulas

$$\psi_i = \alpha_i \left[\int_{R_0}^{R_0+h} + k \int_{R_\alpha}^{R_0} \{[r/v(r)]^2 - \alpha_i^2\}^{-1/2} \right] \frac{dr}{r}, \tag{8a}$$

where α_i are raypath parameters; $k=0$ for upgoing rays; $k=2$ for downgoing rays; R_α is the distance from the Earth's center to the deepest point along the path; h is a hypocentral depth measured from the ground surface. Obviously, $h = R_3 - R_0$, where R_3 is the Earth's radius in the epicenter.

Determination of ψ_i from (8) requires knowledge of α_i and h. The formula for determining the travel time of a seismic wave is:

$$t_i = \left[\int_{R_0}^{R_0+h} + k \int_{R_\alpha}^{R_0} \right] \frac{[r/v(r)]}{([r/v(r)]^2 - \alpha_i^2)^{1/2}} \frac{dr}{r}. \tag{8b}$$

With the raypath parameters given by the Bendorf formula $\alpha_i = v^{-1}(R_3)\sin\beta$, where β_i is the angle of emergence for the i-th station, h can be easily found from (8b), providing the observed times $\tilde{t}_i$ and the times given by (8b) do not differ much.

Because the angles of emergence are usually determined rather inaccurately, they are assumed to be unknown. In that case the ray parameters α_i are given by (8b), while the ψ_i are functions of hypocentral depth, that is, are not uniquely determined. Hence every r_i in (2), and consequently every $\tilde{t}_i$, has a corresponding set ω of h, d_i. Every set of h, d_i $(i=1, n)$ will have a corresponding solution $\{X, Y, Z\}$ of (4): their totality generates a set Ω whose elements are X_0, Y_0, Z_0.

If the origin time is known, then the ray parameters h, and hence Δ_h, can be found by trying values of h over a grid α_i and assuming the closeness of the observed travel times $\tilde{t}_i$ and the theoretical times t_i as given by (8b). Finally, we select those values of h and ψ_i which minimize (7).

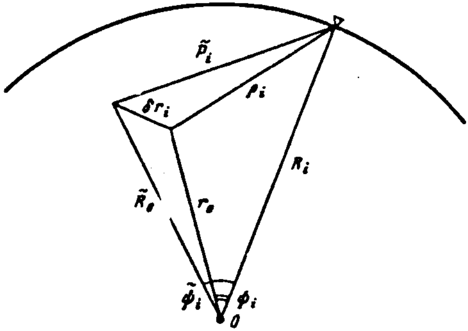

Figure 2 Illustration to inequality (9).

As mentioned above, the usual practice of hypocenter location is to take for the hypocenter that point in ω which minimizes (1).

Let ρ_i, d_i and h correspond to theoretical travel times t_i for the i-th station, where $\rho_i = v_i t_i$ are hypocentral distances; d_i are epicentral distances; h is hypocentral depth; $P_i = u_i \hat{t}_i$, D_i; H being the same quantities, but corresponding to the true hypocenter ($u_i = v_i - \delta v_i$). For the functional S_t we can then write

$$S_t = \sum_{i=1}^{n} (t - \hat{t}_i - \delta t_i)^2 = \sum_{i=1}^{n} v_i^{-2} (v_i t_i - v_i \hat{t}_i - v_i \delta t_i)^2 =$$

$$= \sum_{i=1}^{n} v_i^{-2} (v_i t_i - u_i \hat{t}_i - \hat{t}_i \delta t_i - v_i \delta t_i)^2 = \sum_{i=1}^{n} v_i^{-2} (P_i - \delta P_i - \rho_i)^2.$$

Here $\hat{t}_i = \tilde{t}_i - \delta t_i$; $\delta P = \hat{t}_i \delta v_i + v_i \delta t_i$. Further (Figure 2) we write

$$\begin{aligned} S_t &= \sum_{i=1}^{n} v_i^{-2} (P_i - \delta P_i - \rho_i)^2 = \sum_{i=1}^{n} v_i^{-2} (\tilde{P}_i - \rho_i)^2 \leq \\ &\leq \sum_{i=1}^{n} v_i^{-2} \delta r_i^2 \leq \sum_{i=1}^{n} v_i^{-2} \{ R_i^2 (\bar{\Psi}_i - \psi_i)^2 \} = \\ &= \sum_{i=1}^{n} \nu_i (\tilde{D}_i - d_i) + \Upsilon (\tilde{H} - h)^2 = S, \end{aligned} \tag{9}$$

where $\delta \mathbf{r} = \mathbf{R}_0 - \mathbf{r}_0 = \mathbf{P}_i - \rho_i$; d_i, h and $\tilde{D}_i$, $\tilde{H}$ are epicentral distances and depths corresponding to theoretical and observed travel times; $\nu_i = v_i^{-2}$ and $Y = \sum_{i=1}^{n} \nu_i$ are weights that characterize the inhomogeneity of the earth. It follows from the above relation that the smallness of S_t does not guarantee the smallness of the residual functional in determining the hypocentral depth and epicentral distances, but the smallness of S does involve the smallness of the functional containing time residuals. The above statement follows from the fact that $(\tilde{P}_i - \rho_i)^2$ is the square of differences between the absolute values of $\mathbf{P}_i$ and ρ_i and is independent of their directions, whereas $(\tilde{D}_i - d_i)^2 + (\tilde{H} - h)^2$ is the square of the absolute value of the difference $\mathbf{P}_i - \rho_i$.

Write S in the form

$$S = \sum_{i=1}^{n} \omega_i (\tilde{D}_i - d_i)^2 + (\tilde{H} - h)^2 = \sum_{i=1}^{n} \omega_i R_i^2 (\bar{\Psi}_i - \psi_i)^2 + (\tilde{R}_0 - r_0)^2, \tag{9a}$$

where $\omega_i = \nu_i / \Upsilon$, and formulate Problem 1.

Let $\mathcal{H}$ be a set of potential hypocentral depths, h being an element of that set ($h \in \mathcal{H}$). It is required to find $h^* \in \mathcal{H}$ such as would minimize (9a) under the constraint that X, Y, and Z are determined by solving (5).

This is the problem of hypocenter location, i.e., determining φ, λ and H. However, when dealing with teleseismic events one is more interested in finding four hypocentral parameters: φ and λ, H and τ_0. In that case several approaches are possible.

When the angles of emergence are found from the observations, equations (8b) can be used to determine the hypocentral depth and the origin time τ_0, which equals the arrival time of a seismic wave less the associated travel time.

When we know only arrival times and station coordinates, the problem can be solved as follows. We can write

$$r_i^2 = v_i^2(\tau_i - \tau_0)^2 = v_i^2\tau_i^2 - 2v_i^2\tau_i\tau_0 + v_i^2\tau_0^2,$$

where τ_i are arrival times; τ_0 is the origin time; u_i are effective seismic velocities equal to the ratios of hypocentral distances to travel times.

Substituting the expression for r_i^2 into the right-hand side of (4) and grouping the terms, we have

$$X_0x_i + Y_0y_i + Z_0z_i - T_0\tau_iv_i^2 = 0.5[r_0^2 + R_i^2 - v_i^2(\tau_i^2 + \tau_0^2)]$$

Some more transformations yield the final result

$$Uu_i + Vv_i + Ww_i - Qq_i = \cos(d_i/R_i) - \tau_0\tau_iv_i^2/(r_0R_i), \tag{10}$$

where U, V, W, u_i, v_i, w_i are same as in (5); $Q=T_0/r_0$, $q_1=\tau_iv_i^2/R_i$. Relation (6) must hold as before.

Also, we write the functional

$$\begin{aligned} S &= \sum_{i=1}^{n}\omega_i(D_i - d_i)^2 + (H - h)^2 + \eta(T_0 - \tau_0)^2 = \\ &= \sum_{i=1}^{n}\omega_iR_i^2(\Psi_i - \psi_i)^2 + (R_0 - r_0)^2 + \eta(T_0 - \tau_0)^2, \end{aligned} \tag{11}$$

where $\eta=1/T$; $H=R_3 - R_0$.

We thus arrive at Problem 2. Let $\mathcal{H}$ be a set of possible hypocentral depths, h being an element of that set ($h \in \mathcal{H}$) and $\mathfrak{T}$, a set of possible origin times, τ_0 being an element of that set ($\tau_0 \in \mathfrak{T}$). It is required to find $h^* \in \mathcal{H}$ and $\tau_0^* \in \mathfrak{T}$ that would minimize (11) under the constraint that X, Y, Z and T_0 are found from linear algebraic equations (10). We can now consider the accuracy of teleseismic hypocenter location and discuss the stability of (5) and (10) solution. These aspects were handled earlier [1], [2], [3] for near earthquakes and can be treated accordingly in the present case.

Accuracy of Teleseismic Hypocenter Location. We can write equations (5) and (10) in the matrix form:

$$Kp = f, \tag{12}$$

where $p=\{p_j\}$ are the parameters sought; $j=1, 2, 3$ or $j=1, 2, 3, 4$; K are the matrices of the systems ; $f=\{f_i\}$, $i=1,\dots, n$ the observational data.

If K^TK is a regular matrix, the solution of (12) is given by $p=K^+$, where $K^+=(K^TK)^{-1}K^T$, the superscript ◀t▶ denoting a transpose.

If the vector of the free terms and the matrix K are subject to the uncertainties $\Delta f\neq 0$ and $\Delta K\neq 0$, then the uncertainty of **p** can be found from [3]

$$\tilde{K}\Delta p = \Delta f - \Delta Kp, \tag{13}$$

whose solution is

$$\Delta p = \tilde{K}^+(\Delta f - \Delta Kp).$$

The uncertainties of components Δp_j of $\Delta \mathbf{p}$ are obtained from

$$\Delta p_j = \tilde{K}_j^{+}(\Delta f - \Delta K p),$$

where $\tilde{K}_j^{(+)}$ is a row vector of $\tilde{K}^{+}$.

The absolute value of the j-th component of $\Delta \mathbf{p}$ obeys the inequality

$$|\Delta p_j| = |\tilde{K}_j^{(+)}(\Delta f - \Delta K p)| \leq \|\tilde{K}_j^{(+)}\| \, \|\Delta f - \Delta K p\|, \tag{14}$$

where $\| * \|$ is the euclidean norm. The uncertainty of the total vector is

$$\|\Delta p\| \leq \|\tilde{K}^{+}\| \, \|\Delta f - \Delta K p\|. \tag{14a}$$

Now consider the set of linear equations (5). We assume the uncertainties of the elements of K and of the right-hand sides to be only due to arrival time uncertainties, whose absolute values can be taken equal to $|\delta\tau_i| = \Phi_i|\Delta\tau|$. The weight Φ_i characterizes the measurement quality at the i-th station and a systematic error due to departures of the true velocities from the model.

If $|\delta\tau_i|$ for every $i=1, 2,\ldots, n$ is the mean absolute uncertainty of τ_i, that is,

$$\Phi_i|\Delta\tau| = \mathscr{E}(|\delta\tau_i|),$$

then $\Phi_i = \mathscr{E}(|\delta\tau_i|)/|\delta\tau|$, $|\Delta\tau| \neq 0$.

The normalizing factor $|\Delta\tau|$ can be set equal to the mean absolute uncertainty at all stations, that is, $|\Delta\tau| = \mathscr{E}[\mathscr{E}(|\delta\tau_i|)]$. In that case

$$\Phi_i = \frac{\mathscr{E}(|\delta\tau_i|)}{\mathscr{E}[\mathscr{E}(|\delta\tau_i|)]}.$$

We can put $|\Delta\tau|$ equal to the absolute value of a maximum permissible error for arrival times. Then $0 \leq \Phi_i \leq 1$.

If $|\delta\tau_i|$ is the maximum uncertainty of τ_i at every station ($i=1, 2,\ldots, n$), then

$$\Phi_i = |\delta\tau_i|/|\Delta\tau|,$$

and, if $|\delta\tau_i| = |\Delta\tau|$ ($i=1, 2,\ldots, n$), that is, if $|\delta\tau_i|$ is a maximum uncertainty, then $\Phi_i = 1$.

Apart from random errors in the determination of arrival times, the quantities $|\delta\tau_i|$, and hence Φ_i, may reflect the departures of observed travel times in a real earth from those in an assumed velocity structure.

The weights Φ_i thus reflect both the varying precision of station measurements and the systematic deviations of τ_i due to inhomogeneities of the real earth.

In the case of (5) we have $\Delta K=0$ and $\Delta f \neq 0$. The elements of Δf equal $\delta f_i = -\sin(d_i/R_i)\delta d_i$. Assuming $\delta d_i = \delta\tau_i/\alpha_i$, where $\alpha_i = t_i'$ is the derivative of the travel time curve at the appropriate point, we have

$$\delta f_i = -\sin(d_i/R_i)\delta\tau_i/\alpha_i \quad (\alpha_i \neq 0).$$

The total error vector for (5) is

$$\|\Delta p\| \leq \|\tilde{K}^{+}\| \, \|\sin(d/R)\,\Phi/\alpha\| \, \|\Delta\tau\|. \tag{15}$$

The parameters p_j are in this case equal to $p_1 = X_0/r_0$, $p_2 = Y_0/r_0$, $p_3 = Z_0/r_0$.

Let the set of linear equations (13) be applied to the case in which the unknowns include T_0, in addition to X_0, Y_0, and Z_0. We then have $\Delta K \neq 0$ and $\Delta \neq 0$ with $\delta f_i = -\sin(d_i/R_i)\delta\tau_i/\alpha_i - \delta t_i v_i^2/(r_0 R_i)$ and $(\Delta Kp)_i = -\delta\tau_i T_0 v_i^2/(r_0 R_i)$. Putting $\tau_0 \approx T_0$, we get $(\Delta f - \Delta Kp)_i = -\sin(d_i/R_i)\delta\tau_i/\alpha_i$. It thus appears that we arrive at (15) again with $p_4 = T_0/r_0$.

Let us consider some properties of accuracy estimates (15). In the first place, the estimates are not prone to improvement. This means, in particular, that for any fixed parameters in the right-hand sides of the inequalities one can always choose the signs of arrival time errors so that equalities will be attained. Another property is uniformity. This helps develop a unified approach to the investigation of (5) and (10) for different data sets and hence for different sets of desired parameters. The most important property is that the estimates take account of the uncertainties of K itself, which is essential when the uncertainties are to be estimated in the case of unknown λ, φ, H and T_0.

Let arrival time errors $\delta\tau_i$ be independent variables whose random components have zero means and bounded variances $\sigma_i^2(\Delta f - \Delta Kp)$. In that case the variance of the parameters sought is given by

$$D(\Delta p) = [\tilde{K}^T D^{-1}(\Delta f - \Delta \tilde{K} p)]^{-1},$$

where $D(\Delta p)$ is the covariance matrix of the parameters, $D(\Delta f - \Delta Kp)$ is that of the column vector of free terms. Since the arrival time errors $\delta\tau_i$ are independent, $D(\Delta f - \Delta Kp)$ is a diagonal matrix with the elements $\sin^2(d_i/R_i)\sigma_i^2/\alpha_i^2$; σ_i^2 being the variances of $\delta\tau_i$. Accordingly, the elements of $D^{-1}(\Delta f - \Delta Kp)$ are $[\sin^2(d_i/R_i)\sigma_i^2/\alpha_i^2]^{-1}$.

We have obtained error estimates for the unknown parameters U, V, W and Q. Now we are going to derive estimates for φ, λ, H and T_0. We find $\sigma_T^2 = \sigma_Q^2/r_0^2$ for T_0. Here, σ_T^2 and σ_Q^2 are the variances of T_0 and Q, respectively.

Recalling that $U = \cos\varphi\cos\lambda$, $V = \cos\varphi\sin\lambda$ and $W = \sin\varphi$ and taking the total differential of the right- and left-hand sides of these equalities, we get a set of three linear algebraic equations in two unknowns for determining $\delta\varphi$ and $\delta\lambda$:

$$\begin{aligned} -\sin\varphi\cos\lambda\,\delta\varphi - \cos\varphi\sin\lambda\,\delta\lambda &= \delta U, \\ -\sin\varphi\sin\lambda\,\delta\varphi + \cos\varphi\cos\lambda\,\delta\lambda &= \delta V, \\ \cos\varphi\,\delta\varphi &= \delta W. \end{aligned} \tag{16}$$

The elements of the matrix of the normal equations and of the vector of free terms for the equations are

$$\begin{aligned} &a_{11} = 1; \quad a_{12} = a_{21} = 0; \quad a_{22} = \cos^2\varphi; \\ &a_{13} = \cos\delta W - \sin\varphi(\cos\lambda\,\delta U + \sin\lambda\,\delta V); \\ &a_{23} = \cos\varphi(\cos\lambda\,\delta V - \sin\lambda\,\delta U). \end{aligned}$$

It follows that the errors of φ and λ are

$$\begin{aligned} &\delta\varphi = \cos\varphi\,\delta W - \sin\varphi(\cos\lambda\,\delta U - \sin\lambda\,\delta V); \\ &\delta\lambda = (\cos\lambda\,\delta V - \sin\lambda\,\delta U)/\cos\varphi. \end{aligned}$$

The variances σ_φ^2 and σ_λ^2 of $\delta\varphi$ and $\delta\lambda$ are the diagonal elements of the covariance matrix $D(\delta\varphi, \delta\lambda)$, which is generally not a diagonal one and equals

$$D(\delta\varphi, \delta\lambda) = [K^T D^{-1}(\delta U, \delta V, \delta W) K]^{-1}.$$

Here, K is the matrix of (16); $D^{-1}(\delta U, \delta V, \delta W)$ is the dispersion matrix of δU, δV, and δW.

In order to find the uncertainty of hypocentral depth H location, we need to know the error of R_0. It is easy to verify that (the subscript ≮0≯ is omitted for simplicity)

$$\delta R = U\delta X + V\delta Y + W\delta Z,$$
$$\delta X = r\delta U,\ \delta Y = r\delta V,\ \delta Z = r\delta W.$$

Consequently,

$$\delta R = r(U\delta U + V\delta V + W\delta W).$$

Hence the rms error of δR is

$$\sigma_R = r(U^2\sigma_U^2 + V^2\sigma_V^2 + W^2\sigma_W^2)^{1/2};$$
$$\sigma_H = \sigma_R.$$

Stability of Teleseismic Hypocenter Location and Optimal Network Geometry. The upper bound (14a) provides a guaranteed accuracy of teleseismic hypocenter location. The coordinates of stations can be modified to minimize (14a). Because the right-hand side of (14a) is the estimate of maximum error we arrive at a minimax problem of optimizing the location of seismograph stations over the globe. It can thus be treated as a problem of minimizing the objective function

$$J = \|\tilde{K}^+\|\ \|\sin(d/R)\Phi/\alpha\|,$$

or, to be more precise, some functional of J [3].

The general features of an optimal seismic network can be revealed by examining the objective function J in detail. We represent J as a product of two functions, $J_1 = \|\tilde{K}^+\|$ and $J_2 = \|\sin(d/R)\Phi/\alpha\|$, and consider J_1.

Let K be a full-rank matrix. Then the following bounds hold for $\|\tilde{K}^+\|$ [1]:

$$\left[\sum_{j=1}^{m}\frac{1}{\|\tilde{k}_j\|^2}\right]^{1/2} \le \|\tilde{K}^+\| \le \left[\frac{\prod_{j=1}^{m}\|\tilde{k}_j\|^2}{\det(\tilde{K}^T K)}\sum_{j=1}^{m}\frac{1}{\|\tilde{k}_j\|^2}\right]^{1/2}.$$

The degree of the $\tilde{K}$ stipulation equal to cond $(\tilde{K}) = \|\tilde{K}\|\ \|\tilde{K}^+\|$, can be estimated as

$$\mu(\tilde{K}) \le cond(\tilde{K}) \le \mu(\tilde{K})/\delta(\tilde{K}),$$

where

$$\mu(K) = \left[\sum_{j=1}^{m}\frac{1}{\|\tilde{k}_j\|^2}\sum_{j=1}^{m}\|\tilde{k}\|^2\right]^{1/2};$$
$$\delta(\tilde{K}) = \left[\frac{\det(\tilde{K}^T\tilde{K})}{\prod_{j=1}^{m}\|\tilde{k}_j\|^2}\right]^{1/2}.$$

Consider $\delta(K)$ and $\mu(K)$. The quantity $\delta(K)$ characterizes the skewness of K and has a simple geometrical meaning: δK is the ratio of the volume of the parallelepiped stretched over the column vectors of $\boldsymbol{K}$ to the volume of a rectangular parallelepiped with sides of the same length. It is obvious that $0 \leq \delta(\boldsymbol{K}) \leq 1$, and the equality $\delta(\boldsymbol{K})=1$ is attained when the column vectors of $\boldsymbol{K}$ are mutually orthogonal. When $\delta(\boldsymbol{K})=0$, the column vectors are linearly dependent, and $\|\boldsymbol{K}^+\|$ becomes indefinitely large and the set of linear algebraic equations (SLAE), degenerate.

With $\mu(\boldsymbol{K}) \geq m$, the $\boldsymbol{K}$ matrix has unequal columns. The equality $\mu(\boldsymbol{K})=m$ is attained when the norms of the matrix columns are equal. If the length of a column vector tends to zero, the quantity $\mu(\boldsymbol{K})$ and so $\|\boldsymbol{K}^+\|$ tend to infinity. In that case the SLAE becomes degenerate, too.

The SLAE has the most stable solution, when the determinant of the normal equations is the maximum at a fixed norm of the matrix of the original equations [1]. In this case the degree of matrix stipulation is maximum too. If we consider the columns of a $n \times m$ matrix as n-vectors, then the maximum determinant of the normal equations has corresponding to it the maximum volume of the m-dimensional parallelepiped stretched over the column vectors.

Let us turn to system (5) and write a matrix of normal equations for it:

$$B = K^T K = \begin{bmatrix} \sum_{i=1}^{n} u_i^2 & \sum_{i=1}^{n} u_i v_i & \sum_{i=1}^{n} u_i w_i \\ \sum_{i=1}^{n} u_i v_i & \sum_{i=1}^{n} v_i^2 & \sum_{i=1}^{n} v_i w_i \\ \sum_{i=1}^{n} u_i v_i & \sum_{i=1}^{n} v_i w_i & \sum_{i=1}^{n} W_i^2 \end{bmatrix} \tag{17}$$

According to the Binet-Cauchy formula [6], the determinant of $\boldsymbol{B}$ is

$$\det(B) = \sum_{i_1 < i_2 < i_3}^{n} \begin{vmatrix} u_{i_1} & v_{i_1} & w_{i_1} \\ u_{i_2} & v_{i_2} & w_{i_2} \\ u_{i_3} & v_{i_3} & w_{i_3} \end{vmatrix}^2 . \tag{18}$$

The value of every determinant to be summed in (18) equals six times the volume of the tetrahedron with vertices $M_{i_1}(u_{i_1}, v_{i_1}, w_{i_1})$, $M_{i_2}(u_{i_2}, v_{i_2}, w_{i_2})$, and $M_{i_3}(u_{i_3}, v_{i_3}, w_{i_3})$, lying on a sphere of unit radius, the vertex $M_0(0, 0, 0)$ being at the center of the sphere. The determinant of (17) equals 36 times the sum of squares of the volumes of all tetrahedra with vertices $M_0, M_{i_1}, M_{i_2}, M_{i_3}$.

Naturally, if all vertices (except that at the center of the sphere) lie on the great circle, the volumes of the relevant tetrahedra are zero. This means that, when all observing stations are on the great circle, the determinant of the normal equations is zero, and the equations are degenerate.

The determinant of (17) attains the maximum, when the sum of squares of the volumes of all tetrahedra inscribed in the unit sphere is the maximum. Every set of points on the sphere is a set of vertices of a polyhedron whose volume is the sum of the volumes of the tetrahedra. We know [5] that the polyhedron whose volume is the largest among all polyhedra with a fixed number of vertices inscribed in a fixed smooth convex surface is necessarily a true polyhedron, one whose faces are all triangles lying in different planes. The numerical values of the station coordinates corresponding to the polyhedra of maximum volume are obtained by differentiating the determinant of the normal equations with respect to the coordinates, equating the derivatives to zero, and solving the resulting equations.

Now consider the function $J_3 = \| \sin(d/R)\Phi/\alpha \|$. It is obvious that the objective function J decreases with decreasing J_2. With a fixed hypocenter, J_2 tends to zero as d_i does. With fixed station positions, J_2 is a function of hypocentral coordinates. We need to find the minimum of J_2 as a function of hypocentral coordinates. To do this, we write J_2 in the form

$$J_2 = \left\{ \sum_{i=1}^{n} [1 - (Uu_i + Vv_i + Ww_i)^2]\, \Phi_i/\alpha_i \right\}^{1/2}.$$

Differentiating this expression with respect to U, V, W and equating the derivatives to zero, we get linear equations for determining the coordinates of an optimal hypocenter:

$$U\sum_{i=1}^{n} u_i^2 \frac{\Phi_i}{\alpha_i} + V\sum_{i=1}^{n} u_i v_i \frac{\Phi_i}{\alpha_i} + W\sum_{i=1}^{n} u_i w_i \frac{\Phi_i}{\alpha_i} = \sum_{i=1}^{n} u_i \frac{\Phi_i}{\alpha_i},$$

$$U\sum_{i=1}^{n} u_i v_i \frac{\Phi_i}{\alpha_i} + V\sum_{i=1}^{n} v_i^2 \frac{\Phi_i}{\alpha_i} + W\sum_{i=1}^{n} v_i w_i \frac{\Phi_i}{\alpha_i} = \sum_{i=1}^{n} v_i \frac{\Phi_i}{\alpha_i},$$

$$U\sum_{i=1}^{n} u_i w_i \frac{\Phi_i}{\alpha_i} + V\sum_{i=1}^{n} w_i v_i \frac{\Phi_i}{\alpha_i} + W\sum_{i=1}^{n} w_i^2 \frac{\Phi_i}{\alpha_i} = \sum_{i=1}^{n} w_i \frac{\Phi_i}{\alpha_i}.$$

Note that the matrix of this system differs from that of the normal equations for (5) only by the factors Φ_i/α_i, hence the above reasoning holds true in this case, too.

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