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Optimum arrangement of seismic stations over the Earth's surface

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Abstract. A system of linear algebraic equations was obtained relating hypocenter coordinates and seismic station coordinates on the surface of Earth's ellipsoid. Starting from this system we derived errors in the determination of the ellipsoid parameters and considered the problem of optimum disposition of observation points over the globe surface. Formulated are the main principles for constructing an optimum system of teleseismic observations on the Earth's surface.

Introduction

Up to the present time, the problem of optimum arrangement of seismic stations over the globe's surface has yet not been studied, at least to the author's knowledge. However, being taken from the viewpoint of design experiment theory, the modern teleseismic net is not optimal and does not provide to a great extent for solving a large circle of problems such as localizing hypocenters of small earthquakes ($m \leq 4.5$), monitoring seismicity on all the Earth's surface, studying the deep Earth's structure, and others. Not only does creation of an optimum net of observations allow the problems mentioned to be solved, but it also allows considerable funds to be saved owing to a minimum of observation points.

The present paper presents the system of linear algebraic equations relating the earthquake hypocenter coordinates and seismic station coordinates on the Earth's ellipsoid. Errors in the determining hypocenter parameters were obtained for this system and the problem of optimum array of observation points at the globe surface was considered on the basis of a nonstatistical criterion, the C optimum criterion, earlier introduced by the author. Basic principles were formulated to develop an optimum net of teleseismic observations.

Equations Relating the Coordinates of Hypocenters and Seismic Stations on the Earth's Ellipsoid Surface

Let us write the equations relating the earthquake hypocenter coordinates with the seismic station coordinates. Assume that the geographic coordinates φ_i, λ_i of n seismic stations are given, their elevation (relative to the sea level) being Δh_i, and an earthquake hypocenter has the coordinates φ_0, λ_0 and depth H_0. The origin of the rectangular system of coordinates is placed at the Earth's center. The axis OZ is directed along the polar axis of the Earth's ellipsoid, the axis OX is the crossing of the equator plane and the plane from which latitudes are counted, and the axis OY is in the equator plane and in the meridian whose plane makes the $90°$ angle with the plane of initial meridian. Then we can write the system of nonlinear equations

$$(X_0 - x_i)^2 + (Y_0 - y_i)^2 + (Z_0 - z_i)^2 = r_i^2 \qquad (1)$$

relating the hypocenter coordinates and detecting station coordinates in the rectangular system of coordinate. In (1) X_0, Y_0 and Z_0 are the hypocenter coordinates, x_i, y_i, z_i are the seismic station coordinates, and $\mathbf{r}_i = \mathbf{R}_i - \mathbf{R}_0$; $|\mathbf{R}_i| = R_i = \sqrt{x_i^2 + y_i^2 + z_i^2}$; $|\mathbf{R}_0| = R_0 = \sqrt{X_0^2 + Y_0^2 + Z_0^2}$; $i = 1, 2, \ldots, n$.

The relations between the geographic and Cartesian coordinates on the ellipsoid surface are written as follows [*Zakatov*, 1964]:

$$x = \frac{a \cos\varphi \cos\lambda}{\sqrt{1 - e^2 \sin\varphi}}, \quad y = \frac{a \cos\varphi \sin\lambda}{\sqrt{1 - e^2 \sin\varphi}}$$
$$z = \frac{a(1 - e^2)\sin\varphi}{\sqrt{1 - e^2 \sin\varphi}} \qquad (2)$$

where a is the semimajor axis of ellipsoid, $e^2 = (a^2 - b^2)/a^2$ is the first eccentricity of meridian ellipse, and b is the semiminor axis of ellipsoid.

We seek a solution of (1) on the globe of the radius R_0. In this case the relations are valid

$$X_0 = R_0 \cos\varphi \cos\lambda, Y_0 = R_0 \cos\varphi \sin\lambda, Z_0 = R_0 \sin\varphi \qquad (2')$$

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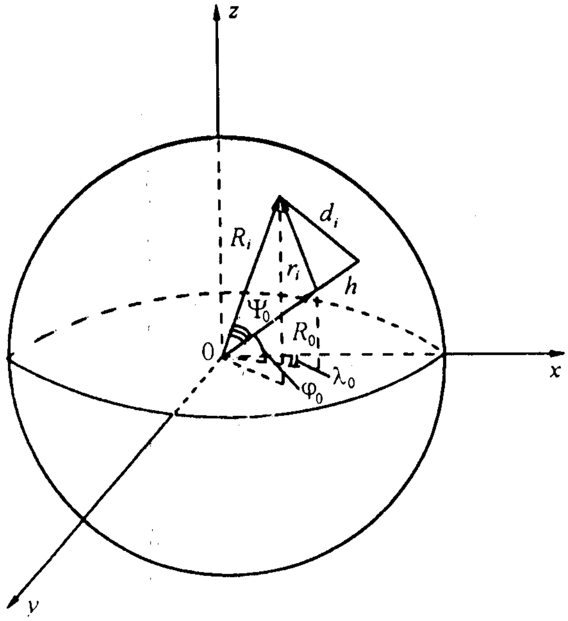

Figure 1. Explanation of the relation between the geographic and geocentric coordinates.

Let us remove the parenthesis in (1) and make use of the relations for R_i and R_0 (see Figure 1). Then, we have

$$X_0 x_i + Y_0 y_i + Z_0 z_i = 0.5(R_0^2 + R_i^2 - r_i^2) \qquad (3)$$

The right-hand side of (3) involves the values r_i^2 equal to

$$r_i^2 = R_0^2 + R_i^2 - 2R_0 R_i \cos\psi_i$$

where $\psi_i = d_i/R_i$ is the angle between $\mathbf{R}_0$ and $\mathbf{R}_i$, and d_i is the epicentral distance.

Let us denote $r_0 = R_0$. Substituting ψ_i defined above into the expressions for r_i^2 and then r_i^2 into (3), we divide the left and the right parts by $r_0 R_i$ and find

$$U u_i + V v_i + W w_i = \cos(d_i/R_i) = \cos(\psi_i) \qquad (4)$$

where $U = X_0/r_0$; $V = Y_0/r_0$; $W = Z_0/r_0$; $u = x_i/R_i$; $v_i = y_i/R_i$; $w_i = z_i/R_i$; $r_0 = R_E - h$; h is the hypocenter depth counted from the Earth's surface, R_E is the Earth's radius at a considered surface point which equals $\sqrt{X^2 + Y^2 + Z^2}$, and X, Y, Z are determined from (2). Also, the hypocenter coordinates are not to be out the Earth's ellipsoid, i.e., the sum of the squares of the unknowns U, V, and W is to satisfy the inequality

$$U^2 + V^2 + W^2 \le \frac{1 + e^4 \sin^2\varphi - 2e^2(\sin\varphi)/a}{1 - e^2 \sin\varphi} \qquad (5)$$

Equation (4) represents itself the system of linear algebraic equations relative to the three unknown quantities U, V, and W. Involved in the right-hand side of (4) are the angle epicenter distances ψ_i on the surface of the sphere with the radius R_i; these distances depend on h and are to be preliminarily calculated. Solving (4) we obtain

$$X_0 = U/r_0; Y_0 = V/r_0; Z_0 = W/r_0$$

$$R_0 = \sqrt{X_0^2 + Y_0^2 + Z_0^2}; H = R_E - R_0$$

Thus we considered the equations relating the hypocenter coordinates φ, λ, and H and the coordinates of seismic stations situated on the Earth's ellipsoid. In practice, on localizing distant earthquakes, of great interest is, however, the problem of determining the four hypocenter parameters φ, λ, H and τ_0, i.e., properly the hypocenter coordinates and occurrence earthquake time or the origin time. For such a case we write the quantities r_i^2 in the form

$$r_i^2 = c_i^2(\tau_i - \tau_0)^2 = c_i^2\tau_i^2 - 2c_i^2\tau_i\tau_0 + c_i^2\tau_0^2$$

where τ_i is the arrival time of seismic waves at a station, τ_0 is the origin time, and c_i are effective seismic wave velocities defined as the ratio of the hypocenter distance to propagation time of seismic wave along ray.

Let us substitute the derived expression for r_i^2 in the right-hand side of (3) and group together the terms. As a result, we have

$$X_0 x_i + Y_0 y_i + Z_0 z_i - T_0 \tau_i c_i^2 =$$
$$= 0.5[r_0^2 + R_i^2 - c_i^2(\tau_i^2 + \tau_0^2)]$$

or, after the relevant transformations, the final relation

$$U u_i + V v_i + W w_i - Q q_i =$$
$$= \cos(d_i/R_i) - \tau_0 \tau_i c_i^2/(r_0 R_i) \qquad (6)$$

where U, V, W, u_i, v_i, w_i are the same quantities as in (4), $Q = \tau_0/r_0$ and $q_i = \tau_i c_i^2/R_i$. Here, as it was above, (5) holds.

Let us consider the error estimations in determining hypocenter parameters for far earthquakes and on the questions of stability of solutions being derived from the system of linear equations (4) and (6). These questions in the case of near earthquakes were studied earlier [*Burmin*, 1976, 1986]. The results of these works are used below.

Estimates of Errors of Hypocenter Parameters for Far Earthquakes

Let us write (4) and (6) in the matrix form

$$Kp = f \qquad (7)$$

re $p = \{p_j\}$ are the unknown parameters; $j = 1, 2, 3$ $j = 1, 2, 3, 4$; K are the matrices of the systems; $\{f_i\}$ are the observed values, and $i = 1, \ldots, n$.

roviding that matrix $K^T K$ is not degenerate the ution of (7) is given by

$$p = K^+ f$$

ere $K^+ = (K^T K)^{-1} K^T$ and the superscript T indites transposition.

In the case when the free term vector f and the matrix are given with the errors $\Delta f \neq 0$ and $\Delta K \neq 0$, we ve the equation for the error of the vector p [*Burmin*, 986]:

$$\tilde{K} \Delta p = \Delta f - \Delta K p \qquad (8)$$

The solution of this equation is

$$\Delta p = \tilde{K}^+ (\Delta f - \Delta K p)$$

The errors of the individual components Δp_j and of he vector Δp satisfy the next relation

$$\Delta p_j = \tilde{k}^{(+)} (\Delta f - \Delta K p)$$

in which $\tilde{k}^{(+)}$ is the row vector of the $\tilde{K}^+$ matrix.

For the absolute value of the jth component of the Δp vector the inequality took place

$$|\Delta p_j| = |\tilde{k}^{(+)} (\Delta f - \Delta K p)| \leq \|\tilde{k}_j^{(+)}\| \, \|\Delta f - \Delta K p\| \qquad (9)$$

ith the Euclidean norm $\|{*}\|$. Correspondingly, we have r the total vector Δp

$$\|\Delta p\| \leq \|\tilde{K}^+\| \, \|\Delta f - \Delta K p\| \qquad (9')$$

Consider now the sys em of equation (4). We assume at the errors in both the elements of the matrix K and in the right-hand [illegible] by errors in the arrival times of waves τ_i whose absolute values may be taken as $|\delta\tau_i| = \rho_i |\Delta\tau|$. The weight factor ρ_i characterizes an equality of measurements at the ith station as well as a systematic error caused by a deviation of the true velocities from an accepted model.

If $|\delta\tau_i|$ for each $i = 1, 2, \ldots, n$ is the average value of the absolute error in the determined τ_i, i.e.,

$$\rho_i |\Delta\tau_i| = \mathcal{E}(|\delta\tau_i|) \ldots \rho_i \ldots |\delta\tau_i| \text{ [illegible]}$$

hen

$$|\Delta\tau| \neq 0$$

The normalizing factor $|\Delta\tau|$ may be put to equal an verage value of the absolute error at all seismic stations articipating in an observation, i.e., $|\Delta\tau| = \mathcal{E}[\mathcal{E}(|\delta\tau_i|)]$. this case --

$$\rho_i = \frac{\mathcal{E}(|\delta\tau_i|)}{\mathcal{E}[\mathcal{E}(|\delta\tau_i|)]}$$

It may be assumed for $|\Delta\tau|$ to be equal to the absolute value of maximum admissible error in the determined arrival times of seismic waves. Then, $0 \leq \rho_i \leq 1$.

If $|\delta\tau_i|$ is the maximum value of an error in the determined τ_i at every of stations ($i = 1, 2, \ldots, n$), then

$$\rho_i = \frac{|\delta\tau_i|}{|\Delta\tau|}$$

and if $|\delta\tau_i| = |\Delta\tau|$ ($i = 1, 2, \ldots, n$), i.e., $|\delta\tau_i|$ is the maximum value of error, then $\rho_i = 1$.

Except for random errors in the arrival times of seismic waves, the values $|\delta\tau_i|$ and thereby ρ_i may reflect deviations of the observed travel times of waves propagating in the real inhomogeneous three-dimensional medium from the travel times of waves propagating in a model medium with an assumed velocity distribution.

Thus the weight factors ρ_i account for both station measurements of varying accuracy and systematic deviations in the observed τ_i associated with an inhomogeneity of the real medium.

Let us turn to (4). Here, we have $\Delta K = 0$, and $\Delta f \neq 0$. In the given case the elements of the vectors Δf equal $\delta f_i = -\sin(d_i/R_i)\delta d_i$. Considering $\delta d_i = \delta\tau_i/\alpha_i$, where $\alpha_i = t'_i$ is the derivative of the travel time curve at the corresponding point, we find $\delta f_i = -\sin(d_i/R_i)\delta\tau_i/\alpha_i (\alpha_i \neq 0)$. The total error vector or (4) is to satisfy the inequality

$$\|\Delta p\| \leq \|\tilde{K}^+\| \, \|\sin(d/R)\rho/\alpha\| \, |\Delta\tau| \qquad (10)$$

The parameters p_j in the case considered are consequently equal to $p_1 = X_0/r_0$, $p_2 = Y_0/r_0$, and $p_3 = Z_0/r_0$.

Let (8) be referred to the case when along with X_0, [illegible] value of T_0 is also unknown. For [illegible] and Δ [illegible] $-\sin(d_i/R_i)\delta\tau_i/\alpha_i - \delta\tau_i\tau_0 c_i^2/(r_0 R_i)$ and $(\Delta K p)_i = -\delta\tau_i T_0 c_i^2/(r_0 R_i)$. Using $\tau_0 \approx T_0$ gives $(\Delta f - \Delta K p)_i = -\sin(d_i/R_i)\delta\tau_i/\alpha_i$. Thus in this case too we arrive at the inequality (10), and $p_4 = T_0/r_0$.

Some properties of the estimate (10) are worth to be noted. First, these estimates cannot be improved [illegible] means that the [illegible] reach the equalities. Another feature of the estimates is their uniformity. Such a property allows a unified approach to be developed in studying (4) and (6) for various initial data sets and correspondin[illegible] various sets of the unknown parameters. At last, the property that appears to be the most important is that these estimates account for the errors in the matrix K itself, which plays the basic role in estimating the errors for the case with the unknowns λ, φ, H, and T_0.

Let the errors $\delta\tau_i$ in the arrival times be independent values having random components with nonzero expectations and the finite dispersions $\sigma_i^2(\Delta f - \Delta Kp)$. Then, the dispersion is known to be determined from the relation [*Hudson*, 1970]

$$D(\Delta p) = [\tilde{K}^T D^{-1}(\Delta f - \Delta Kp)\tilde{K}]^{-1}$$

where $D(\Delta p)$ is the covariant matrix of the estimated parameters, and $D(\Delta f - \Delta Kp)$ is the covariant matrix of the column vector of the free terms in the linear system. Because the errors $\delta\tau_i$ in the arrival times are independent values, the matrix $D(\Delta f - \Delta Kp)$ is the diagonal one with elements to equal $\sin^2(d_i/R_i)\sigma_i^2/\alpha_i^2$ (σ_i^2 is the dispersion of the values $\delta\tau_i$). Therefore the matrix $D^{-1}(\Delta f - \Delta Kp)$ elements are equal to $1/[\sin^2(d_i/R_i)\sigma_i^2/\alpha_i^2]$.

Obtained above were the estimates of the errors in the inferred unknown parameters U, V, W, and Q. Let us now find the estimates for φ, λ, H, and T_0. We derive immediately $\sigma_T^2 = \sigma_Q^2/r_0^2$ for T_0. Here, σ_T^2 and σ_Q^2 are the dispersions of the parameters T_0 and Q, respectively.

Taking into account $U = \cos\varphi\cos\lambda$, $V = \cos\varphi\sin\lambda$ and $W = \sin\varphi$ and forming the total increment of the right and left parts of these equalities, we find the system of the three linear equations to relate the two unknowns $\delta\varphi$ and $\delta\lambda$

$$\left.\begin{array}{c} -\sin\varphi\cos\lambda\,\delta\varphi - \cos\varphi\sin\lambda\,\delta\lambda = \delta U \\ -\sin\varphi\sin\lambda\,\delta\varphi + \cos\varphi\cos\lambda\,\delta\lambda = \delta V \\ \cos\varphi\,\delta\varphi = \delta W \end{array}\right\}$$

The matrix elements of normal equations and of the free term vector for this system are

$$a_{11} = 1; a_{12} = a_{21} = 0; a_{22} = \cos^2\varphi$$

$$a_{13} = \cos\varphi\,\delta W - \sin\varphi(\cos\lambda\,\delta U + \sin\lambda\,\delta V)$$

$$a_{23} = \cos\varphi(\cos\lambda\,\delta V - \sin\lambda\,\delta U)$$

Therefore the errors in φ and λ equal to

$$\delta\varphi = \cos\varphi\,\delta W - \sin\varphi(\cos\lambda\,\delta U - \sin\lambda\,\delta V)$$

$$\delta\lambda = (\cos\lambda\,\delta V - \sin\lambda\,\delta U)/\cos\varphi$$

The dispersions σ_φ^2 and σ_λ^2 of the quantities $\delta\varphi$ and $\delta\lambda$ are to equal to the values of the diagonal elements of the covariant matrix $D(\delta\varphi, \delta\lambda)$ which is generally not diagonal and equal

$$D(\delta\varphi, \delta\lambda) = [K^T D^{-1}(\delta U, \delta V, \delta W)K]^{-1}$$

Here, K is the matrix of (11), and $D^{-1}(\delta U, \delta V, \delta W)$ is the dispersion matrix of the parameters δU, δV, and δW.

Finding the errors in the determined hypocenter depth H requires the error in R_0 to be known. It is easily seen that (for the sake of simplicity we omit here the subscript "0")

$$\delta R = U\delta X + V\delta Y + W\delta Z$$

and

$$\delta X = r\delta U, \quad \delta Y = r\delta V, \quad \delta Z = r\delta W$$

Therefore

$$\delta R = r(U\delta U + V\delta V + W\delta W)$$

Hence the root-mean-square deviation of the parameter δR is to equal

$$\sigma_R = r\sqrt{U^2\sigma_U^2 + V^2\sigma_V^2 + W^2\sigma_W^2}$$

and $\sigma_H = \sigma_R$.

Stability of Determination of the Hypocenter Parameters for Distant Earthquakes

The estimate (9′) is majorant providing the guaranteed accuracy for the hypocenter parameters of remote earthquakes. By varying the observation point coordinates one can choose their arrangement in such a manner as to minimize (9′). Since the right-hand side of (9′) represents itself the estimate of maximum error in the hypocenter parameters, we arrive at the minimax problem to determine an optimal arrangement of seismic stations over the terrestrial globe. Thus the problem to find an optimal geometry of the observation system may be viewed as the minimization problem with the objective function

$$J = \|\tilde{K}^+\|\,\|\sin(d/R)\rho/\alpha\|$$

or more precisely, with some functional of the objective function J [*Burmin*, 1986].

To bring out the most general features of the optimal system of seismological observations, we must consider the objective function J in detail. Let us represent J in the form of product of the two functions $J_1 = \|\tilde{K}^+\|$ and $J_2 = \|\sin(d/R)\rho/\alpha\|$ and consider each of them separately.

First, we turn to the function $J_2 = \|\sin(d/R)\rho/\alpha\|$. It is understandable that if the objective function is the smaller, the smaller are the values of the J_2 function. As the hypocenter position is fixed and J_2 approaches zero, d_i is tending to zero as well. For a fixed position of the hypocenter J_2 is the function of hypocenter coordinates. To find the minimum of J_2 as the function of hypocenter coordinates we write J_2 in the form

$$J_2 = \left\{ \sum_{i=1}^{n} [1 - (Uu_i + Vv_i + Ww_i)^2] \rho_i^2/\alpha_i^2 \right\}^{\frac{1}{2}}$$

Differentiating the latter expression with respect to U, V, and W and setting the derivatives equal to zero, we obtain the system of linear equations determining the optimum hypocenter coordinates

$$\begin{aligned}
U\sum_{i=1}^{n} u_i^2 \frac{\rho_i^2}{\alpha_i^2} + V\sum_{i=1}^{n} u_i v_i \frac{\rho_i^2}{\alpha_i^2} + W\sum_{i=1}^{n} u_i w_i \frac{\rho_i^2}{\alpha_i^2} &= \sum_{i=1}^{n} u_i \frac{\rho_i^2}{\alpha_i^2} \\
U\sum_{i=1}^{n} u_i v_i \frac{\rho_i^2}{\alpha_i^2} + V\sum_{i=1}^{n} v_i^2 \frac{\rho_i^2}{\alpha_i^2} + W\sum_{i=1}^{n} v_i w_i \frac{\rho_i^2}{\alpha_i^2} &= \sum_{i=1}^{n} v_i \frac{\rho_i^2}{\alpha_i^2} \\
U\sum_{i=1}^{n} u_i w_i \frac{\rho_i^2}{\alpha_i^2} + V\sum_{i=1}^{n} w_i v_i \frac{\rho_i^2}{\alpha_i^2} + W\sum_{i=1}^{n} w_i^2 \frac{\rho_i^2}{\alpha_i^2} &= \sum_{i=1}^{n} w_i \frac{\rho_i^2}{\alpha_i^2}
\end{aligned} \tag{11}$$

We note that the matrix of this system differs from the matrix of the normal equations for (4) only by the factors ρ_i^2/α_i^2. The solution of (11) determines a point within the globe and is mostly dependent upon the observation point coordinates.

Now, we consider function J_1. Let the matrix $\tilde{K}$ be the one of full rank. Then, the two-sided estimate holds true for $||\tilde{K}^+||$ [*Burmin*, 1976]:

$$\left(\sum_{j=1}^{m} \frac{1}{||\tilde{k}_j||^2} \right)^{\frac{1}{2}} \leq ||\tilde{K}^+|| \leq \left(\frac{\prod_{j=1}^{m} ||\tilde{k}_j||^2}{\det(\tilde{K}^T\tilde{K})} \sum_{j=1}^{m} \frac{1}{||\tilde{k}_j||^2} \right)^{\frac{1}{2}}$$

The relevant estimate of the condition number $\tilde{K}$ to be equal to $\mathrm{cond}(\tilde{K}) = ||\tilde{K}|| \, ||\tilde{K}^+||$ has the form

$$\mu(\tilde{K}) \leq \mathrm{cond}(\tilde{K}) \leq \frac{\mu(\tilde{K})}{\delta(\tilde{K})}$$

where

$$\mu(K) = \left(\sum_{j=1}^{m} \frac{1}{||\tilde{k}_j||^2} \sum_{j=1}^{m} ||\tilde{k}_j||^2 \right)^{\frac{1}{2}}$$

$$\delta(\tilde{K}) = \left(\frac{\det(\tilde{K}^T\tilde{K})}{\prod_{j=1}^{m} ||\tilde{k}_j||^2} \right)^{\frac{1}{2}}$$

Consider the values $\delta(K)$ and $\mu(K)$. The first characterizes a skewness of the matrix K and has a simple geometrical meaning, $\delta(K)$ in fact being the ratio of the parallelepiped volume stretched on the column vectors of the matrix K to the volume of rectangular parallelepiped with edges of the same length. It is evident that $0 \leq \delta(K) \leq 1$, and the equality $\delta(K) = 1$ is reached only in the case when the column vectors of the matrix K are mutually orthogonal. If $\delta(K) = 0$, then the column vectors are linearly independent, and the value of $||K^+||$ tends to infinity. Then, the system of linear equations becomes degenerate.

We have the value $\mu(K) \geq m$ characterizing the weight distribution over the matrix K columns. Also, the equality $\mu(K) = m$ is reached when the column norms are equal to each other, i.e., for the matrices equally weighted on their columns. If the length of one of the column vectors of the matrix K tends to zero, then the value of $\mu(K)$ and consequently $||K^+||$ are both tending to infinity. In this case the system of linear equations becomes degenerate.

This system has the most stable solution when the determinant of the normal system of equations takes a maximum at the fixed norm of the matrix of the initial system of equations [*Burmin*, 1976]. In such a case the condition number also takes a maximum. If one considers columns of a $n \times m$ matrix as vectors in a n-dimensional space, then to the maximum value of the determinant of the normal system there corresponds the maximum volume of m-dimensional parallelepiped stretched on the matrix column vectors.

An Optimum Geometry of Teleseismic Observation Network

Let us turn to (4). We have for it the matrix of the normal equations

$$B = K^T K = \begin{pmatrix} \sum_{i=1}^{n} u_i^2 & \sum_{i=1}^{n} u_i v_i & \sum_{i=1}^{n} u_i w_i \\ \sum_{i=1}^{n} u_i v_i & \sum_{i=1}^{n} v_i^2 & \sum_{i=1}^{n} v_i w_i \\ \sum_{i=1}^{n} u_i w_i & \sum_{i=1}^{n} v_i w_i & \sum_{i=1}^{n} w_i^2 \end{pmatrix} \tag{12}$$

According to the Binet-Cauchy formula [*Faddeyev and Faddeyeva*, 1963], the determinant of the B matrix is

$$\det(B) = \sum_{i_1<i_2<i_3}^{n} \begin{vmatrix} u_{i_1} & v_{i_1} & w_{i_1} \\ u_{i_2} & v_{i_2} & w_{i_2} \\ u_{i_3} & v_{i_3} & w_{i_3} \end{vmatrix}^2 \tag{13}$$

The value of each determinant standing in the summands of (13) is independent upon the observed hypocenter position and equal to the sextuple volume of the tetrahedron with apices $M_{i_1}(u_{i_1}, v_{i_1}, w_{i_1})$, $M_{i_2}(u_{i_2}, v_{i_2}, w_{i_2})$, and $M_{i_3}(u_{i_3}, v_{i_3}, w_{i_3})$ situated on the unit sphere, the apex $M_0(0,0,0)$ being at the center of this sphere. Thus the determinant of the matrix (12) of the normal equations is composed of 36 values of the sum of squares of all tetrahedron volumes with apices M_0, M_{i_1}, M_{i_2}, and M_{i_3}.

It is clear that if all the apices of tetrahedrons (excepting that at the center of the sphere) lie in the same arc of a great circle then the volumes of correspond-ing tetrahedrons are coincident with each other. This implies that as all observation points lie in one arc of a great circle, the determinant of the system of normal equations equals zero, and the system itself is degenerate.

Consider now (6). We have for it the relation similar to (13):

$$\det(B) = \sum_{i_1<i_2<i_3}^{n} \begin{vmatrix} u_{i_1} & v_{i_1} & w_{i_1} & -q_{i_1} \\ u_{i_2} & v_{i_2} & w_{i_2} & -q_{i_2} \\ u_{i_3} & v_{i_3} & w_{i_3} & -q_{i_3} \\ u_{i_4} & v_{i_4} & w_{i_4} & -q_{i_4} \end{vmatrix}^2 \tag{14}$$

Let us decompose the determinants in (13) with respect to the last column. Then, we find

$$\det(B) = \sum_{i_1<i_2<i_3}^{n} \left\{ q_{i_1} \begin{vmatrix} u_{i_2} & v_{i_2} & w_{i_2} \\ u_{i_3} & v_{i_3} & w_{i_3} \\ u_{i_4} & v_{i_4} & w_{i_4} \end{vmatrix} - q_{i_2} \begin{vmatrix} u_{i_1} & v_{i_1} & w_{i_1} \\ u_{i_3} & v_{i_3} & w_{i_3} \\ u_{i_4} & v_{i_4} & w_{i_4} \end{vmatrix} + \right.$$
$$\left. + q_{i_3} \begin{vmatrix} u_{i_1} & v_{i_1} & w_{i_1} \\ u_{i_2} & v_{i_2} & w_{i_2} \\ u_{i_4} & v_{i_4} & w_{i_4} \end{vmatrix} - q_{i_4} \begin{vmatrix} u_{i_1} & v_{i_1} & w_{i_1} \\ u_{i_2} & v_{i_2} & w_{i_2} \\ u_{i_3} & v_{i_3} & w_{i_3} \end{vmatrix} \right\}^2 \tag{15}$$

It is easy to verify that all the determinants in braces are equal to the sextuple volumes of the tetrahedrons with apices M_{i_2}, M_{i_3}, M_{i_4}, M_0; M_{i_1}, M_{i_3}, M_{i_4}, M_0; M_{i_1}, M_{i_2}, M_{i_4}, M_0; M_{i_1}, M_{i_2}, M_{i_3}, M_0 and do not depend on the hypocenter position, whereas the value of the determinant of the matrix B itself is dependent on the hypocenter position. Nevertheless, the greater is the value of the determinant of the B matrix for every fixed hypocenter, the greater is the value of each determinant in (15), i.e., the greater are the volumes of all the tetrahedra with the common apex M_0. Thus the optimum arrangement of seismic stations on the terrestrial globe is weakly dependent upon hypocenter position and is mainly determined by an internal geometry of observation network.

To obtain numerical values of the coordinates of observation points corresponding to the maximum of the determinant of the normal system, one has to differentiate the determinant with respect to the coordinates, to equate the derivatives to zero and then to solve the derived system of equations.

At the same time, (13) and (14) take the maximum values in the case when the sum of squares of the volumes of all tetrahedrons inscribed in the unit sphere becomes maximal. However, in every set of the points on a sphere is that of apices of a polyhedron the ume of which equals to the sum of volumes of rele tetrahedrons with one common apex at the center of sphere. It is known [*Feysh Tot,* 1958] that a polyhed having the greatest volume among all the polyhed inscribed in a smooth convex surface is necessarily correct polyhedron with triangular faces, i.e., has o the triangular faces lying in different planes. In parti lar, the volume of polyhedron inscribed in the unit sol sphere is subject to the condition [*Feysh Tot,* 1958]

$$V \le \frac{2k}{3}\cos^2\frac{\pi f}{2k}\cot\frac{\pi e}{2k}\left(1-\cot^2\frac{\pi f}{2k}\cot^2\frac{\pi e}{2k}\right)$$

where e, f, and k the numbers of apices, faces, and edg of a convex polyhedron. The equality is reached here f a polyhedron with triangular faces. Three such polyh drons are known to exist: tetrahedron actahedron, a icosahedron [*Feysh Tot,* 1958].

If any earthquake were detected at any point of t terrestrial point, then to obtain the optimal observati system for localizing an earthquake hypocenter it wou suffice to place seismic stations at the apices of c of regular tetrahedrons inscribed in the Earth's glol However, because earthquakes of different magnitu are detected at different distances from a hypocent the distances between stations of an observation net to be chosen in accord with a minimum power of earthquake to be detected. Let us consider this qu tion in more detail.

The earthquake magnitude by definition is [*Kond skaya,* 1981]

$$m = \log(A/T) + \delta(\Delta, h, s(T,\omega)) - \Delta m$$

where A is the maximum amplitude of ground displa ment expressed in microns (μ), at a given station and a considered type of seismic wave, T is the period of wave, $\delta(\Delta, h, s(T,\omega))$ is the so-called calibrating cu (function) for a region under question, and Δm is correction accounting for a systematic deviation of magnitude estimated at the station. When the m nitude of an earthquake is too small, remote stat or stations with a small amplification will not re the earthquake. Therefore to select the correct bas observation network we need to know the distribu of minimum values $m_{\min}$ of the magnitudes dete by a given seismic station with the epicentral dista *Antonova et al.* [1974] suggested the following for for $m_{\min}$:

$$m_{\min} = \log(\gamma A_n/(VT)) + \delta(\Delta, h, s(T,\omega)) - \Delta r$$

Here, the parameter γ is equal to the smallest adm ble ratio of a desired signal amplitude to the noise sufficient to observe the signal. (In practice, it is generally assumed that $\gamma \sim 1.5$, A_n is the noise amplitude on seismogram in millimeters, and V is the instrument amplification in thousands of times.) The amplification is commonly chosen so as to have the 1-mm value for A_n.

The calibrating function $\delta(\Delta, h, s(T, \omega))$ has been sufficiently studied. The values of the calibrating function for various types of instruments and for various region of the Community of Independent States involving the near and far wave zones are given in the instruction on making and processing observations at seismic stations of the united network "ESSN SSSR" [*Nauka*, 1981]. The values of Δm for various regions are presented, for example, in the work by ***Antonova et al.*** [1974]. The absolute value of the correction Δm is usually not more than 0.25.

Figure 2 shows the dependence of the magnitude $m_{\min}$ on the epicentral distance for a seismic station with the amplification of $V = 40{,}000$ ($\Delta m = 0$). In this case the averaged calibrating curve has been used [*Nauka*, 1981].

Thus the optimal net of observations is given by the apices of regular or semiregular polyhedron having (1) triangles as its faces and (2) the length of face side which is close to the given value, in accord with the minimal magnitude of recorded earthquake so as to detect the earthquake at least by three (four) stations. To construct such a system we may take one of the regular polyhedrons (tetrahedron, octahedron or icosahedron) inscribed in the terrestrial globe. It is known that a tetrahedron has 4 faces, 4 apices and 6 edges, an octahedron has 8 faces, 6 apices, and 12 edges, and an icosahedron has 20 faces, 12 apices, and 30 edges. Let us project every of the three polyhedrons from the globe center onto the Earth's surface. Then, we obtain the rectangular spherical polyhedrons whose faces are equilateral spherical triangles. The length of each triangle side is $\Delta = \arctan 2$ expressed in radians [*Volynskiy*, 1977] and $l = R \arctan 2$ expressed in kilometers.

An icosahedron among the rectangular polyhedrons with triangular faces has the maximum number of apices. The length of icosahedron edge inscribed in the Earth's sphere equals $l \approx 7076$ km. In line with the curve in Figure 2, such an observation system will detect earthquakes with the magnitude of ~5 and more by help of three and a greater number of stations. To obtain the system with smaller minimum bases, let us divide the spherical triangles into four parts in the next manner. We connect the midpoints of each triangle following along the arc of the great circle. This results in a triangulation of each triangle in the four triangles with sides of ~ 3538 km. Being continued, such a subdivision procedure gives rise to an observation system with even smaller bases. In the general case, an nth multiple

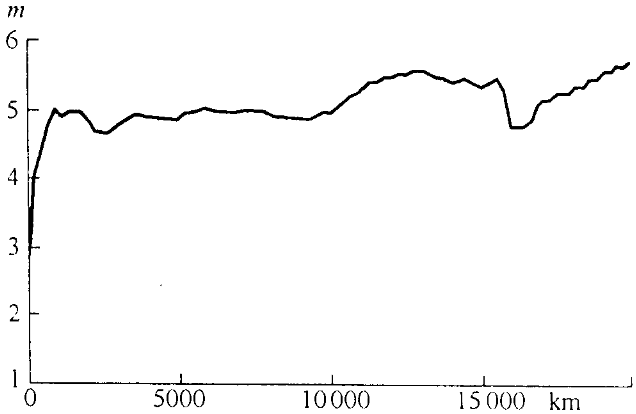

Figure 2. $m_{\min}$ versus the epicentral distance (on the data of *Antonova et al.* [1974]).

partitioning of a spherical polyhedron produced by an icosahedron, we have $f_n = 20 \times 4^n$ faces, $k_n = 30 \times 4^n$ edges, and $e_n = 10 \times 4^n + 2$ apices. The fourth multiple separation results in a system with bases of ~ 442 km having 2562 observation points. Figure 3 shows the orthogonal projection of hemisphere with such a system onto the plane. The system obtained will record without gaps earthquakes of magnitude ~ 4 when using instrument amplification 40,000, and earthquakes of magnitude ~ 3.5 when amplification is 200,000. When the Earth is considered as a sol' ' sphere, the points of such a system represent themselves the apices of a polyhedron such that six triangles meet at the apices, except for the 12 apices being the apices of an initial icosahedron to which five isosceles triangles are brought. However, when taking into account the Farth's ellipticity the polyhedron produced by this constructing would have all the faces different from those of isosceles triangles but would nevertheless remain the true polyhedron.

The system described is clear to be not uniquely defined, but with accuracy to an arbitrary turn about any of three mutually perpendicular and arbitrarily chosen axes passing through the solid sphere center. The system is appropriate to be chosen in such a way as to have the greatest number of its points coinciding with available observation points or being at least close to them. Keeping this in mind, let us consider the functional

$$L = \sum_{i=1}^{n} \sum_{j=1}^{m} [(\varphi_i - \varphi_j)^2 + (\lambda_i - \lambda_j)^2 + \varepsilon_i]^{-1}$$

where φ_i and λ_i are the coordinates of existing observation points, n is the number of these points, φ_j and λ_j are the coordinates of planned seismic stations, m is the number of these stations, and ε_i are the posi-

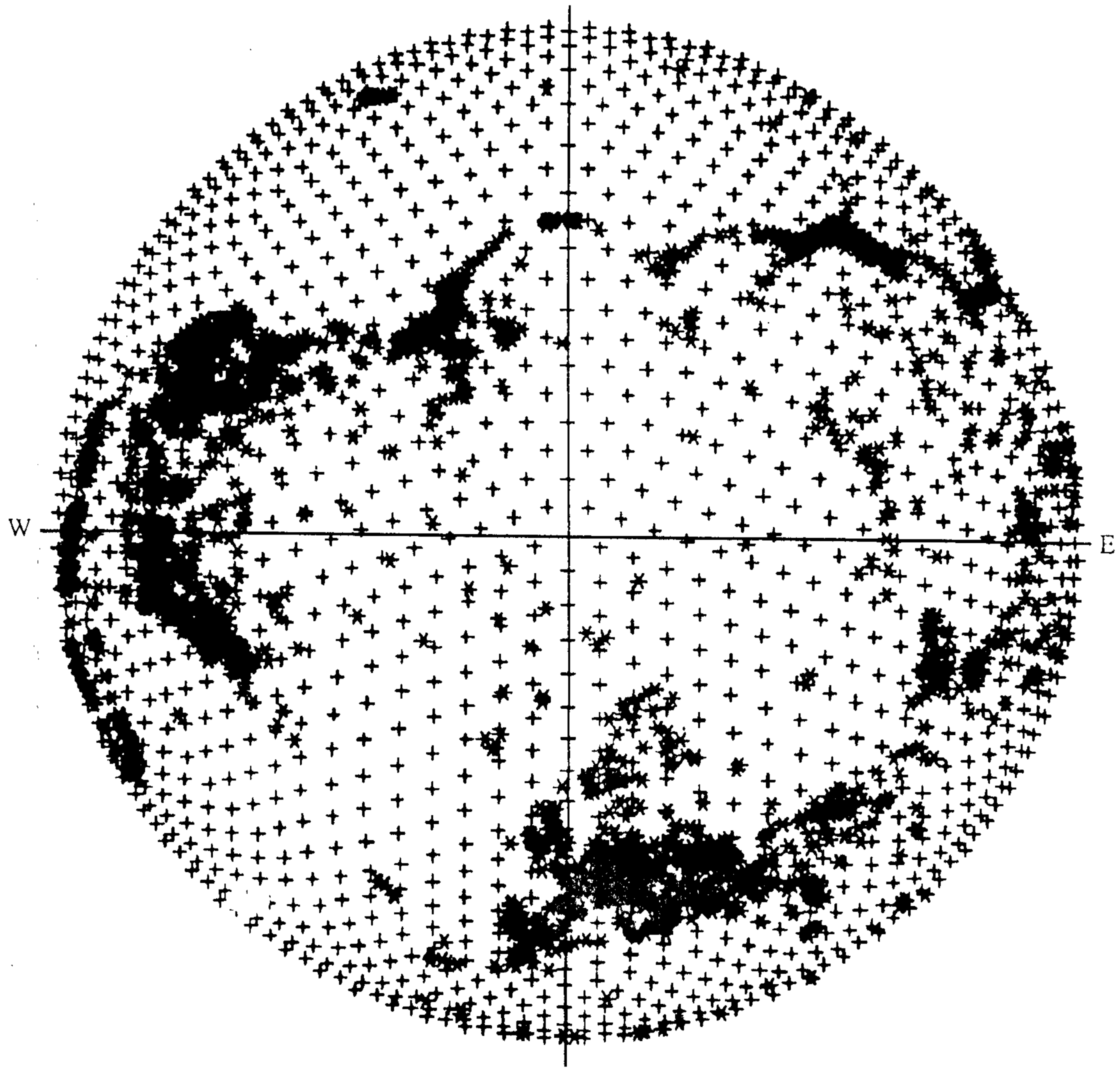

Figure 3. Orthogonal projection of unit hemisphere with observation points onto the plane. The projection center is coincident with the apice of regular icosahedron. Plus signs show points of the optimum teleseismic observation net, and asterisks show points of now existing observation net.

tive numbers ($\varepsilon_i > 0$). The problem of determining the coordinates φ_i and λ_i consists in the L functional minimization, the summation being allowed to be held only over the indices of neighboring points. The values of ε_i are chosen to give preference to the points $M_i = \{\bar{\varphi}_i, \bar{\lambda}_i\}$ and $M_j = \{\varphi_j, \lambda_j\}$ coinciding with one another within a given accuracy. This accuracy also defines the ε_i values.

To find the maximum of the functional L we may employ the exhaustion procedure, applying it to the points from the range θ of the Earth's surface which is outlined by some solid angle Ω. The angle Ω is taken on the basis of the following arguments. Any operating seismic station falls into one of the spherical triangles resultant from the subdivision of spherical icosahedron faces. A circle of the radius of $r \approx l/\sqrt{3}$ may be circumscribed about each of the triangles. This circle cuts in the sphere of a radius R a cone with the apex angle to equal $\alpha = 2\arcsin(r/R)$. Thus all the surface of the globe will be covered when varying the increments of the polar coordinates within the limits $\delta\varphi \in [0, \alpha]$ and $\delta\theta \in [0, \alpha]$.

Conclusions

The optimum systems of teleseismic observations for two cases, when either only the hypocenter coordinates or the latter together with the earthquake occurrence time are determined, are coincident with each other at

$n > 3$ and represent themselves a covering of the terrestrial globe by triangles at the apices of which seismic stations are to be situated. These triangles involving almost all the globe's surface form hexagons at the apices and in the center of which observation points are located, and in the 12 ranges these triangles constitute pentagons with their centers being coincident with the apices of the initial icosahedron. At the same time the optimum seismological network in the plane also represents itself also a covering with hexagons, with observation points at their centers [*Burmin and Akhmet'ev*, 1994]. Thus the optimum seismic observation network on the globe and in the plane are coincident for almost all of the Earth except for 12 ranges. The sizes of these ranges depend on the partition multiplicity of the initial icosahedron. The more the multiplicity, the smaller are these ranges. If an octahedron or tetrahedron were taken as the initial regular polyhedron, then the number of such regions would be six and four, respectively, but the triangles in them would produce quadrangles and triangles, correspondingly. It is clear that projecting the optimum observation system is desired to be designed so as the national and regional nets be included into the global one as their combined part.

The greater part of observation points (some 1700) of the optimum seismological network for the terrestrial globe falls on sea and oceans. The importance of carrying out observations not only on the ground but also at the sea bottom is unquestionable, and technical difficulties related to long-term observations in oceans, excepting possibly the arctic areas, are overcome. In addition to seismological observations, these points may be used in observing other geophysical fields.

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