Izvestiya, Physics of the Solid Earth, Vol. 37, No. 9, 2001, pp. 717–723. Translated from Fizika Zemli, No. 9, 2001, pp. 19–26.

An Algorithm for the Numerical Construction of *C*-Optimal Designs: A Case Study of the Design of Seismological Observations Networks

Fong Nguen Wan and V. Yu. Burmin

Schmidt United Institute of Physics of the Earth, Russian Academy of Sciences, Bol'shaya Gruzinskaya ul. 10, Moscow, 123810 Russia

Received October 12, 1998

Abstract—The MINCOND algorithm is proposed for the numerical construction of discrete *C*-optimal designs. The efficiency of the MINCOND algorithm is illustrated by solving the problem of seismic observation network design. The algorithm allows one to find *C*-optimal designs of a large dimension.

INTRODUCTION

The solution of many seismological problems substantially depends on how well the hypocenter coordinates and earthquake onset times are determined. It is common knowledge that the determination accuracy of these parameters largely depends on the relative position of stations and on the location of earthquake hypocenters. Therefore, the problem of finding an optimal geometry for a seismic network is of great current interest, particularly in regions with complex boundaries, including multiply connected regions. Examples of such regions are the Far East transition zone (Kamchatka, Kurile Islands, and the continental part of Russia's Far East), the insular portion of Southeast Asia, and mountain areas such as the Caucasus, Tien Shan, and Pamirs.

In this work, the design problem of an experiment is solved in relation to the determination of hypocentral parameters of earthquakes recorded by observation systems with a limited number of points. In view of this, this paper only examines discrete designs. In the theory of design of experiment, a design $\xi(N)$ is a set of observation points x_i ($i = 1, 2, \ldots, n$) and the corresponding numbers of observations r_i at these points, $\sum_{i=1}^{n} r_i = N$. If r_i/N can assume any values between 0 and 1, the design is called a continuous design; otherwise, it is discrete [Fedorov, 1971]. The essence of the problem of constructing a discrete optimal design is the search for a global extremum of a certain target function or functional, specifically related to a chosen optimality criterion. In problems of the optimal design of experiment, the choice of an experimental quality criterion is a rather difficult task, mainly because none of the available criteria fully reflects all specific features of problems in question.

The available numerical methods of constructing optimal designs are based on a design optimality criterion. In practice, however, the search for a global extremum of an arbitrary functional is rather complicated, which makes most optimization algorithms inefficient: hence the need to find more efficient algorithms and methods for solving such problems in accordance with various design optimality criteria.

In recent years, some researchers have proposed numerical algorithms for constructing discrete optimal designs for various optimality criteria. Such algorithms are mostly special procedures ("exchange algorithms") intended for constructing optimal designs based on the so-called *D*-optimality criterion, implying minimization of the determinant of a pertinent normal system of linear equations (for a review of optimality criteria, see [Fedorov, 1971; Burmin, 1995]). When solving problems of the design of physical experiments in the case if design matrices are perturbed and the error distribution laws are unknown (as is usual in practice), it is appropriate to use a nonstatistical design optimality criterion, namely, the *C*-optimality criterion. This criterion was first proposed by Burmin [1976] in relation to the idea of maximum inversion stability in mathematical physics [Marchuk, 1973; Uspenskii and Fedorov, 1974]. Recently, the *C*-optimality criterion has been successfully applied to problems of experimental design in seismology, thermal physics, etc. [Burmin, 1986, 1994; Alifanov *et al.*, 1988].

In this paper, we propose an algorithm for constructing optimal discrete designs based on the *C*-optimality criterion. This algorithm is a modification of procedures of random global search in the global optimization of multivariable functions and the exchange procedure of constructing optimal designs. The efficiency of the algorithm is confirmed by the results of solving optimal design problems for regional seismological observation networks (e.g., the Armenian network [Avetisyan *et al.*, 1999].

C-OPTIMALITY CRITERION

Let the goal of an experiment be to find estimates of unknown parameters p linearly related to observed values f. This relation can be written in the matrix form:

$$Kp = f, \tag{1}$$

where K is a matrix of full rank.

We assume that the matrix K and the vector of free terms f are specified with errors ΔK and Δf. These errors can be either random or systematic. Then, Δp can be determined using a system of linear algebraic equations that can also be represented in the matrix form [Burmin, 1995]:

$$\tilde{K}\,\Delta p = \Delta f - \Delta K p, \tag{2}$$

where $\tilde{K}$ is the perturbed matrix equal to $\tilde{K} = K + \Delta K$.

The problem of experimental design in this case is to find a configuration of observation points minimizing the norm Δp at given ΔK and Δf.

The solution of Eq. (2) is written as

$$\Delta p = (\tilde{K}^T\tilde{K})^{-1}\tilde{K}^T(\Delta f - \Delta K p) = \tilde{K}^+(\Delta f - \Delta K p).$$

Here, the following estimates are valid:

$$\|\Delta p\| \le \|\tilde{K}^+\|(\|\Delta K\|\|p\| + \|\Delta f\|), \tag{3}$$

$$\frac{\|\Delta p\|}{\|p\|} \le \|\tilde{K}\|\|\tilde{K}^+\|\left(\frac{\|\Delta f\|}{\|\tilde{f}\|} + \frac{\|\Delta K\|}{\|\tilde{K}\|}\right)\frac{\|K\|}{\|\tilde{K}\|}, \tag{4}$$

where the symbol $\|\circ\|$ denotes a generally arbitrary norm of vectors and matrices.

Relations (3) and (4) yield respective estimates of the absolute and relative deviations of the vector p. One can easily see that, with decreasing $\|\tilde{K}^+\|$ and $\|\tilde{K}\|\|\tilde{K}^+\|$, these estimates also decrease.

Thus, the problem of experimental design is advantageously reduced to the minimization of the quantities $\|\tilde{K}^+\|$ or $\|\tilde{K}\|\|\tilde{K}^+\|$. The value $\mathrm{cond}(\tilde{K}) = \|\tilde{K}\|\|\tilde{K}^+\|$ is called the condition number of the matrix $\tilde{K}$. The boundedness of $\mathrm{cond}(\tilde{K})$ is a necessary and sufficient condition for problem (1) to be well-posed, and the optimality criterion related to the minimization of $\|\tilde{K}^+\|$ or $\mathrm{cond}(\tilde{K})$ is called the *C*-optimality criterion [Burmin, 1976, 1995].

Furthermore, it should be noted that all computational properties of most algorithms for solving the system of equations (1) are largely dependent on the operator $\tilde{K}^+$ [Leonov, 1987; Mikhlin, 1988]; therefore, the validity of the *C*-optimality criterion is a necessary and sufficient condition for the solution of system (1) to be stable.

ALGORITHMS OF THE EXCHANGE FOR CONSTRUCTING DISCRETE OPTIMAL DESIGNS

The available algorithms for constructing optimal designs fall into two groups depending on the type of observations. In the first group of algorithms, optimal designs are sought on a limited set of designs, with a finite number of measurements at each observation point (discrete designs). In the second group, optimal designs are sought on sets with an arbitrary number of measurements at observation points (continuous designs). Note that, because efficient algorithms for finding optimal discrete designs are absent, this problem is often solved by using algorithms of the second group, which can only yield a rough approximation.

Now, we discuss some of the well-known algorithms for constructing optimal discrete designs. These algorithms mostly belong to the group of exchange algorithms [Fedorov, 1986]. The best known among them are the following two algorithms.

Fedorov's Algorithm

Let observations in a physical experiment be performed at N points. Let S be a set of discrete designs $\xi(N) \subset S$, consisting of L measurements, or candidate points, representing possible observation points and numbers of measurements at them in this experiment. The number N is much smaller than L: $L \gg N$. The problem is to find an optimal design $\xi^*(N) \subset S$ complying with some specified criterion.

The optimal experimental design is found as follows [Fedorov, 1971]. First, a nondegenerate initial design $\xi_0(N) \subset S$ is determined in an arbitrary way. A design is nondegenerate if the matrix determinant of the related normal system of equations is nonzero. Then, one measurement is transferred from point x_i^+ to another point x_j^- (x_i^+, $x_j^- \subset \xi_0$), and the best design ξ_1 is determined in accordance with the chosen optimality criterion, and so forth. The selection of the pair of points (x_i^+, x_j^-) for updating the current design $\xi(N) \subset S$ is controlled by a certain function related to the covariance matrices of the designs, i.e., the covariance matrices of the estimated parameters [Fedorov, 1971]. This procedure generates a sequence of designs that converges, in accordance with the chosen criterion, to a design that is better than the initial one but, nevertheless, is not necessarily the optimal design.

DETMAX Algorithm

The DETMAX algorithm was developed by Mitchell [1974] on the basis of the algorithm proposed by Wynn [1970] for constructing optimal designs according to the *D*-optimality criterion. This algorithm also is an exchange algorithm and, in a way, resembles

Fedorov's algorithm. First, the search procedure randomly selects a nondegenerate initial design. This design is then successively updated by exchanging points. As distinct from Fedorov's algorithm, variances of estimated errors of the sought-for values play the role of the function controlling the selection of the pair of exchange points. This procedure also generates a sequence of designs converging to a design better than the initial design.

We should note that both algorithms (Fedorov's and Wynn–Mitchell) involve a simultaneous exchange of one pair of points and do not always guarantee convergence to an optimal design.

Modifications of algorithms of the Fedorov's and DETMAX types with the aim of improving convergence and increasing the probability of finding an exact optimal design were proposed in [Atwood, 1973; Wu, 1978; Galil and Kiefer, 1980; Cook and Nachtsheim, 1980; Johnson and Nachtsheim, 1983; Wierich, 1986; Atkinson and Donev, 1988]. A detailed comparison of these algorithms was made by Cook and Nachtsheim [1980] and Fedorov [1986]. Nearly all of these algorithms involve the exchange of a pair of points.

Yonchev [1988] proposed an algorithm called FDOV for constructing *D*-optimal designs; this is a modification of the DETMAX algorithm with the use of a procedure for exchanging a finite number of points $M \geq 1$. This algorithm consists of two stages. The first stage serves to determine a certain subset $P \subset S$ containing M points, where S is the set of L candidate points ($M \leq L$). The subset P is randomly chosen and is included in the initial design $\xi_0(N) \subset S$. The initial design $\xi_0(N)$ is randomly chosen, as in the other algorithms. As a result, we have a new design $\xi_1(N \cup M) \subset S$. At the second stage, one point out of M points is removed at each step from the set $N \cup M$ in such a way that each new design ensures a minimum variance of estimated errors of the sought parameters; and so forth. Thus, this procedure generates a sequence of designs converging to the best design. Yonchev [1988] notes that, in 50% of cases, the FDOV algorithm is more efficient than Fedorov's and DETMAX algorithms. Nevertheless, the FDOV algorithm also fails to find with certainty an optimal design.

The slow convergence and low probability of finding the optimal design, inherent in the existing exchange algorithms, are due to the fact that, at each succeeding step, the information on the candidate points that were used at the preceding stages in the search for the optimal design is ignored. This makes it necessary to seek new, more efficient algorithms for constructing discrete *C*-optimal designs. In this work, we develop a new algorithm of this type and present numerical modeling results showing it to be more efficient than the aforementioned algorithms.

MINCOND ALGORITHM FOR CONSTRUCTING A *C*-OPTIMAL DESIGN

Block Diagram of the Algorithm

The procedure for finding an optimal design in the algorithm proposed here consists of two stages. The purpose of the first stage is to find a *C*-optimal design on the basis of a modified procedure of exchanging a finite number of points, according to which, at each step, the points that participated in the exchange at the preceding steps are excluded from consideration. At the second stage, the optimal design is updated by successively decreasing the dimension of the subset P, or the number of variable exchange points. A block diagram of the algorithm is shown in Fig. 1a.

Now, we describe the algorithm in more detail. Its *first stage* consists of external and internal cycles. The external cycle of the first stage involves the determination of random initial designs (points) covering the entire set of candidate points (SCP), and each succeeding initial design, except the first one, is constructed after the best (quasi-optimal) design was found at the preceding step of the cycle and incorporates the result obtained.

In the internal cycle, a modified exchange procedure described below is performed for each initial covering. First of all, an SCP is specified. The SCP should be a regular grid with its chosen in accordance with the criterion of design discrimination; i.e., after any design point is exchanged for its nearest point, the corresponding designs should be distinguishable. Two designs are regarded as distinguishable, or nonequivalent, if the following relation holds true:

$$|\Phi(\xi_1) - \Phi(\xi_2)| > \sigma, \tag{5}$$

where Φ is the quality criterion of experiment and σ is a given positive value.

We construct a sequence k of the subsets of points $\{P_i\}$ $(i = 1, 2, 3, \ldots, k)$. Let $\xi_i(N)$ denote the design resulting from the exchange at the ith step. Each subset P_i $(P_i \subset S)$ consists of M randomly selected points subjected to the condition $P_i \cap \xi^j_{i-1}(N) = \varnothing$, where $\varnothing$ is the empty set. This means that the points participating in the exchange at the ith step should not coincide with points of the design obtained at the $(i-1)$th step. Two designs are considered noncoincident if they meet the condition $\rho(\xi^{j+1}(N), \xi^j(N)) > \varepsilon$, where ρ characterizes the proximity of candidate points of two designs in a metric defined below and ε is a given positive number. The choice of the number k is constrained by the condition $\bigcup_i^k P_i = S$, meaning that the union of all subsets P_i covers the entire set S. The sequence $\{P_i\}$ generates a nonincreasing sequence $\|K_i^+\|$ $(i = 1, 2, \ldots, k)$ converging in norm to a matrix corresponding to the quasi-optimal design $\xi_k(N)$.

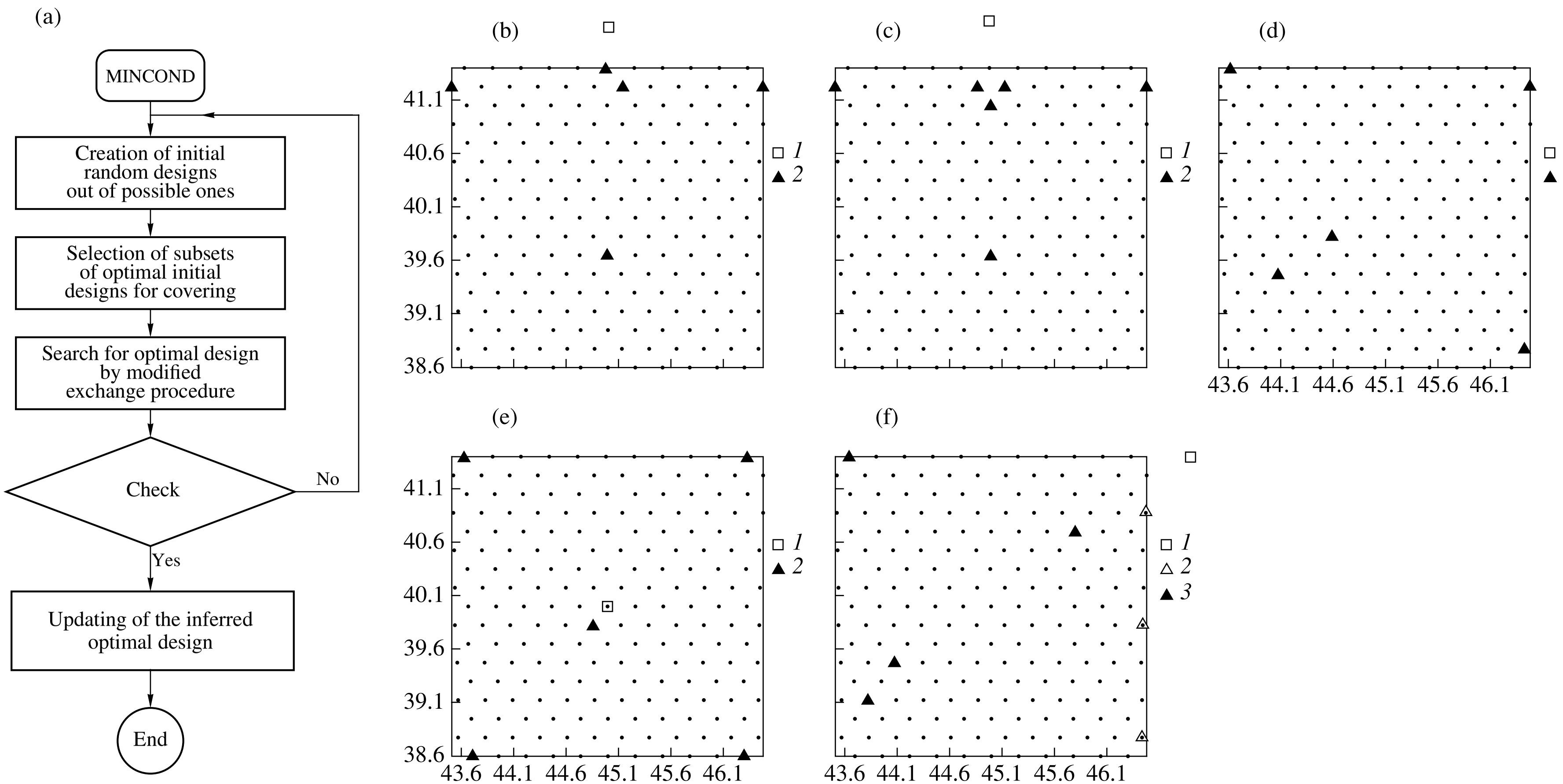

Fig. 1. (a) Block diagram of the algorithm. (b)–(e) Positions of the hypocenter (*1*) and optimal observation system (*2*) consisting of (b, d, e) five and (c) six observation points. (f) Positions of the hypocenter (*1*), existing points (*2*), and additional optimal four-point observation system (*3*).

This procedure is applied to the set of all given random initial designs. In order to increase the probability of finding an optimal design, it is necessary to have a maximum number of noncoincident initial designs covering the entire set S. However, an increase in the number of initial designs substantially increases the computation time because of additional iterations, which is often undesirable. In this case, it is essential to determine a minimum set of initial designs consistent with a maximum efficiency of the search for an optimal design; i.e., the task is to find an optimal set of initial designs. In order to choose an optimal set of initial designs, we apply the following procedure.

Let $\Re(\xi^j(N))$ denote a sequence of exchange operators defined on the set of designs in each internal cycle j, i.e., $\Re(\xi^{j-1}(N)) = \xi^j(N)$, and let $\xi_k^j(N)$ denote a quasi-optimal design obtained upon accomplishing each internal cycle.

We construct a sphere $B(\xi_k^j, \varepsilon)$ of radius ε centered at the quasi-optimal design $\xi_k^j(N)$:

$$B(\xi_k^j, \varepsilon) = \{(\xi^{j+1}(N)) \subset S, \rho(\xi^{j+1}(N), \xi_k^j(N)) \le \varepsilon\}, \tag{6}$$

where ρ is the metric defined as follows:

$$\rho(\xi^{j+1}, \xi_k^j(N)) = \sup(\alpha_1\rho_1 + \alpha_2\rho_2), \xi^{j+1}, \xi_k^j(N) \subset S, \tag{7}$$

Here, $\rho_1 = [\sum_{i=1}^{N}(\varphi_i^{j+1} - \varphi_{ik}^j)^2 + (\lambda_i^{j+1} - \lambda_{ik}^j)^2]^{1/2}$; $\rho_2 = |||K^+(\xi^{j+1}(N)|| - ||K^+(\xi_k^j(N)|||$; $\varphi_i^{j+1}, \lambda_i^{j+1}$; $\varphi_{ik}^j, \lambda_{ik}^j$ are the coordinates of points of the corresponding designs ξ^{j+1} and $-\xi_k^j(N)$ ($i = \overline{1, N}$); and $\alpha_1, \alpha_2 \ge 0$ are given weights.

The external cycle yields a certain set of spheres. We set $Z_r = \bigcup_{j=1}^{r} B(\xi_k^j, \varepsilon)$.

At each iteration of the external cycle, the search for an optimal design starts with an initial design that lies outside the spheres constructed at the previous iterations of the external cycle. During the search, the new initial design is constructed on the set $Q = (S \setminus Z_r)$. The search for an optimal design terminates when $Q = \varnothing$, i.e., when the initial designs exhaust the entire set S.

As a result of the first stage in the search for an optimal design, we obtain a sequence of quasi-optimal designs that contains the best design $\xi^*(N)$ of the entire sequence.

At *the second stage* of the algorithm, a quasi-optimal design is updated by means of the following procedure. We fix one point from $\xi^*(N)$ and seek an optimal design by the exchange procedure described above on condition that the chosen point does not change. As a result, we obtain a new best design $\xi_{1,i}^*(N)$.

Repeating this procedure and fixing successively all remaining $N - 1$ points from $\xi^*(N)$, we obtain a sequence of best designs $\xi_{1,i}^*(N)$, from which we select a new quasi-optimal design $\xi_1^*(N)$.

The entire procedure is then repeated for the design $\xi_1^*(N)$ and with two fixed points, one of which is retained from the preceding procedure and remains unchanged in the search process, and the other is one of the remaining points of the design $\xi_1^*(N)$ and is successively exchanged. As a result of this step, we obtain a new quasi-optimal design $\xi_{2,i}^*(N)$; and so forth.

The entire procedure is repeated until the number of fixed points becomes equal to $N - 1$, or until the designs begin to improve. This procedure generates a sequence of quasi-optimal designs converging in norm to an optimal design.

AN ALGORITHM FOR FINDING THE OPTIMAL GEOMETRY OF A SEISMOLOGICAL OBSERVATION NETWORK

In order to illustrate the efficiency of our algorithm, we address the problem of determining an optimal configuration of seismological observation points when recording near earthquakes. We assume that the Earth's surface is a plane and, hence, the system of equations relating the coordinates of the earthquake source and the coordinates of the recording stations has the form [Burmin, 1995]

$$Xx_i + Yy_i + t_0\nu^2 t_i - 0.5\eta = f_i, \tag{8}$$

where $i = 1, 2, \ldots, n \ge 4$; $f_i = 0.5\,(x_i^2 + y_i^2 - \nu^2 t_i^2)$; $\eta = X^2 + Y^2 + H^2 - \nu^2\tau_0^2$; X, Y, H, and t_0 are the hypocenter coordinates and the earthquake onset time (time at the source); x_i, y_i, and t_i are the coordinates of the seismic stations that recorded the earthquake and the arrival times of seismic waves at these stations ($i = \overline{1, n}$); and ν_i are effective seismic velocities numerically equal to the ratios of the linear distance between the ith station and the hypocenter to the raypath traveltime.

The problem of finding the optimal configuration of seismic observation points reduces to the search for station locations on a plane such that ensures minimum errors and maximum stability of estimates of the solution to (8). According to the C-optimality criterion, this leads to the search for a system of observation points $T^* = \{T_i(x, y)\}$ such that $\|\tilde{K}^+(T^*)\| = \min_{T \subset \Omega}\|\tilde{K}^+(T)\|$, or $T^* = \operatorname{Arg}\min_{T \subset \Omega}\|\tilde{K}^+\|$, where Ω is the experimental design region.

In designing an optimal seismological observation network on a plane, the choice of the SCP plays a substantial role and requires the fulfillment of certain conditions.

First of all, it is necessary to correctly specify the grid type (triangular, square, pentagonal, etc.) Previous results [Burmin, 1995] indicate that, on a plane, an optimal seismological observation network is realized on a grid of equilateral triangles. Thus, in order to determine an optimal seismological observation network, the SCP should be specified on a triangular grid with a step Δ that covers the entire region studied. The grid step is chosen in accordance with criterion (5).

To illustrate the efficiency of the MINCOND algorithm, we consider a few examples of numerical determination of an optimal observation network.

Let the study region be specified by the geographic coordinates $\varphi = 43.5°–46.5°$ and $\lambda = 38.5°–41.5°$. We arbitrarily cover this region by a triangular grid with the step $\Delta = 22.5$ km, consisting of 195 candidate points. Since the hypocenters of earthquakes that occurred in the study region prior to the optimization of the observation network have a certain spatial distribution, we determine the optimal system relative to the center of this distribution [Burmin, 1995]. In all examples, to simplify calculations, we consider a homogeneous medium with the compressional wave velocity $v_p = 6.0$ km/s.

Example 1. The center of the distribution lies at the point with the coordinates $\varphi = 45°$, $\lambda = 41.8°$, $h = 15$ km. The problem is to determine an optimal network consisting of five observation points ($N = 5$).

We set $\sigma \approx 0.05$, $\varepsilon \approx 0.1$, and $\alpha_1 = \alpha_2 \equiv 1$ in expressions (5)–(7). The numerical experiment showed that M (the number of points simultaneously participating in the exchange) should lie within the interval INT(N/2) + $1 \le M \le N$ providing the most rapid convergence to the optimal design. In this case, we set $3 \le M = 4 \le 5$. The result of the search for the optimal geometry of observation points is shown in Fig. 1b.

Example 2. The center of the hypocenter distribution lies at the same point as in Example 1, and the problem is to optimize the system consisting of six stations. As before, M is set equal to 4. Figure 1c shows the optimal configuration of observation points constrained by the solution.

Note that in both cases the observation points nearest to the hypocenter (two and three points, respectively) are located nearly at the same point (a difference of at least one grid spacing in point locations is a necessary condition for the program operation, even if the points coincide). Actually, the optimal observation system in both cases consists of four points, and this complies with general principles of the design theory of seismological observation systems [Burmin, 1995].

Example 3. The center of the earthquake hypocenter distribution lies at the point with the coordinates $\varphi = 46.85°$, $\lambda = 41.4°$, $h = 15$ km. The problem is to find an optimal network of five observation points. The optimal configuration of five observation points is shown in Fig. 1d.

Example 4. The center of the hypocenter distribution coincides with the center of the design region having the coordinates $\varphi = 45°$, $\lambda = 40°$, $h = 15$ km. The problem is to find the optimal geometry of a five-point network. The numerical results are presented in Fig. 1e.

Example 5. The region is assumed to include three existing observation points located at the boundary of the region and having the coordinates ($\varphi_1 = 46.45°$, $\lambda_1 = 38.76°$), ($\lambda_2 = 46.45°$, $\varphi_2 = 39.8°$), and ($\varphi_3 = 46.45°$, $\lambda_3 = 40.87°$); the hypocenter distribution is centered at the point with the coordinates $\varphi = 46.85°$, $\lambda = 41.4°$, $h = 15$ km (Fig. 1f). The problem is to complement, in an optimal way, this system by four observation points. The resultant optimal configuration of the four additional observation points is shown in Fig. 1f.

Rabinowitz and Steinberg [1990] used similar examples to illustrate the results of determining optimal observation systems by the DETMAX algorithm. We should note that their results are inconsistent with optimal designs. This can be attributed to two factors. The first is the poor convergence of the sequence of designs to the optimal design in this algorithm. The second factor is a linearized approach adopted by the authors in their determination of earthquake hypocenter parameters (coordinates and time at the source), which reduces to the Taylor expansion of the residual functional in the neighborhood of the solution. Equations resulting from this linearization are of a local character, and, as a consequence, the problem of observation network design is likewise of local character and its solution largely depends on the initial approximation.

CONCLUSION

The MINCOND algorithm intended for the numerical construction of discrete C-optimal designs is described. This algorithm combines the advantages of the random global search procedure in the global optimization of multivariable functions and the exchange procedure of constructing optimal designs. The results of numerical modeling show that this algorithm is more efficient than similar algorithms based on other optimality criteria and on other strategies of the search for optimal designs.

The results of mathematical modeling presented above were obtained under the assumption that the seismic velocity in the medium is constant. Real media have a more complex structure, with velocities varying both vertically and laterally. However, this fact has virtually no effect on the chosen optimal networks, because the design matrix in this case depends only slightly on the seismic velocity distribution and its properties are controlled solely by the coordinates of observation points. At the same time, this simplification substantially reduces the computation time. This problem was discussed in [Burmin, 1995].

The strategy proposed in this paper for finding the optimal design on the basis of the *C*-optimality criterion can be used for constructing optimal designs on the basis of other design quality criteria. The MINCOND algorithm is applicable to the optimization of observation systems developed for the solution of other geophysical problems, in particular, those related to seismic tomography.

REFERENCES

Alifanov, O.M., Artyukhin, E.A., and Rumyantsev, S.V., *Ekstremal'nye metody resheniya nekorrektnykh zadach* (Extremum Methods of Solving Ill-Posed Problems), Moscow: Nauka, 1988.

Atkinson, A.C. and Donev, A.N., Algorithms, Exact Designs, and Blocking in Response Surface and Mixture Designs, *Optimal Design and Analysis of Experiments*, Holland, 1988.

Atwood, C.L., Sequences Converging to D-Optimal Designs of Experiments, *Ann. Statist.*, 1973, no. 1, pp. 342–352.

Avetisyan, A.M., Burmin, V.Yu., and Nguen Van Fong, Optimal Network of Seismological Observations in Armenia, *Byulleten' stroitelei Armenii*, 1999, no. 4, pp. 41–43.

Burmin, V.Yu., The Experimental Design Problem and the Conditioning of Systems of Linear Algebraic Equations, *Izv. Akad. Nauk SSSR, Ser. Tekhn. Kibern.*, 1976, no. 2, pp. 195–200.

Burmin, V.Yu., *Optimizatsiya seismicheskikh nablyudenii i opredelenie koordinat zemletryasenii* (Optimization of Seismic Observations and Determination of Earthquake Coordinates), Moscow: OIFZ RAN, 1995.

Cook, R.D. and Nachtsheim, C.J., Comparison of Algorithms for Constructing Exact D-Optimal Designs, *Technometrics*, 1980, vol. 2, pp. 315–324.

Fedorov, V.V., The Experimental Design of an Observation Network, *Optimal Algorithms of the Exchange Type*, WP-86-62, IIASA, 1986, pp. 23–30.

Fedorov, V.V., *Teoriya optimal'nogo eksperimenta* (The Theory of Optimal Experiment)), Moscow: Nauka, 1971.

Galil, Z. and Kiefer, J., Time and Space Saving Computer Methods Related to Mitchell's DETMAX for Finding D-Optimum Designs, *Technometrics*, 1980, vol. 22, pp. 301–313.

Johnson, M.E. and Nachtsheim, C.J., D-Optimal Designs on Convex Design Spaces, *Technometrics*, 1983, vol. 25, pp. 271–277.

Leonov, A.S., The Method of Minimum Psuedoinverse Matrix, *Zh. Vychisl. Mat. Mat. Fiz.*, 1987, vol. 27, no. 8, pp. 1123–1138.

Marchuk, G.I., *Metody vychislitel'noi matematiki* (Methods of Computational Mathematics), Novosibirsk: Nauka, 1973.

Mikhlin, S.G., *Nekotorye voprosy teorii pogreshnostei* (Some Problems of the Theory of Errors), Leningrad: LGU, 1988.

Mitchell, J.J., An Algorithm for the Construction of D-Optimal Experimental Designs, *Technometrics*, 1974, vol. 16, pp. 203–310.

Rabinowitz, N. and Steinberg, D.M., Optimal Configuration of a Seismographic Network: A Statistical Approach, *Bull. Seismol. Soc. Am.,* 1990, vol. 80, no. 1, pp. 187–196.

Uspenskii, A.B. and Fedorov, V.V., Design of Experiment in Some Problems of Mathematical Physics, *Kibernetika*, 1974, no. 4, pp. 123–128.

Wierich, W., On Optimal Designs and Complete Class Theorems for Experiments with Continuos and Discrete Factors of Influence, *J. Stat. Plan. Inf.*, 1986, vol. 15, pp. 19–27.

Wu, C.F., Some Iterative Procedures for Generating Nonsingular Optimal Designs, *Commun. Stat.*, 1978, vol. 14, no. 7, pp. 1399–1412.

Wynn, H.P., The Sequential Generation of D-Optimum Experimental Designs, *Ann. Math. Stat.,* 1970, vol. 41, pp. 1655–1664.

Yonchev, H., New Computer Procedures for Constructing Exact D-Optimal Designs, *Optimal Design and Analysis of Experiments*, Holland, 1988, pp. 71–80.