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New Prospects in Interpretation of Converted Seismic Waves

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It is shown that travel times of converted shear and direct compressional waves in a vertically varying elastic earth can be used to find the distributions of shear and compressional velocities simultaneously.

INTRODUCTION

The method of converted earthquake waves is a popular technique used in seismology to find earth structure, especially in the upper layers [2], [3], [5], [9]. Nevertheless, there is no rigorous theory as yet for inverting traveltime differences between compressional and converted shear waves to determine the velocity structure. The conventional use of converted earthquake waves is to determine depths to discontinuities where the conversion takes place, given mean compressional and shear velocities. M. Hasegawa's formulas [3], [9], [10] or other simple relations are commonly used for that purpose. The compressional and converted shear waves corresponding to a single observation site but to different points at the discontinuity of conversion are used to find the depth to the discontinuity, which means that compressional waves travel along different rays to the discontinuity. The fact that the distance between the conversion points remains unknown is a source of uncertainty for finding seismic velocity above the discontinuity.

I propose to determine the velocity and thickness of a layer by using those points in the traveltime curves of compressional and converted shear waves that correspond to the same conversion point and have the same ray parameter, i.e., which travel along the same ray below the conversion point. To be able to deal with this case I derived Fredholm integral equations of the first kind which relate compressional and shear velocities to a traveltime difference between converted shear and direct compressional waves propagating in a vertically varying elastic earth. This gives velocity structure both for compressional and shear waves.

BASIC EQUATIONS

Suppose a source of elastic motion is at a point in an elastic earth. A P wave is assumed to be incident on a discontinuity from below and to excite a converted SV wave which is recorded by seismometers at the ground surface along with the P wave. The assumptions are as follows: (1) the ground surface is plane in the volume where conversion and recording take place; (2) the seismic velocity is bounded on a finite interval of depth [0, z] and is a twice piecewise smooth function of depth $v = v(z)$.

We now write down equations for determining the differences between the respective epicentral distances and travel times for converted SV and P from the conversion discontinuity $z = z^*$ to the recording site in a parametric form

$$\Delta x(p) = \int_0^{z^*} \left\{ \frac{p}{(u_p^2(z) - p^2)^{1/2}} - \frac{p}{(u_s^2(z) - p^2)^{1/2}} \right\} dz, \tag{1}$$

$$\Delta t(p) = \int_0^{z^*} \left\{ \frac{u_s^2(z)}{(u_s^2(z) - p^2)^{1/2}} - \frac{u_p^2(z)}{(u_p^2(z) - p^2)^{1/2}} \right\} dz, \tag{2}$$

where $u_p(z) = v_p^{-1}(z)$ and $u_s(z) = v_s^{-1}(z)$ are reciprocals of P and SV velocities, respectively; p is the ray parameter for P and SV; $0 \le p \le u_p(z^*+0)$. The values of $x_p(p)$, $x_s(p)$ and $t_p(p)$, $t_s(p)$ are assumed to be relevant to one and the same conversion discontinuity, but to different recording sites for direct P and converted SV.

Direct P and converted SV penetrate through the $0z^*$ layer, that is, seismic rays have no turning points in this layer. That means that the travel times and epicentral distances would remain the same, when the $0z^*$ layer is divided into elementary layers, and these are reshuffled. Consequently, the functions $v_p(z)$ and $v_s(z)$ cannot be found from traveltime differences of converted SV and direct P uniquely, because the same travel times of shear $t_s(p)$ and compressional $t_p(p)$ waves correspond to different velocity functions $v_p(z)$ and $v_s(z)$ that have the same measures $H(u)$ and $G(u)$, where [6]

$$H(u) = \text{mes}\{z\colon z \le z^*,\ v_p^{-1}(z) \le u_p\},$$

$$G(u) = \text{mes}\{z\colon z \le z^*,\ v_s^{-1}(z) \le u_s\},$$

The functions $H(u)$ and $G(u)$ have the following properties by definition: (a) they are nondecreasing, (b) vanish when $-\infty < u \le u^* = u(z^*-0)$; (c) are equal to $h = z^*$ when

$u(0) = u_0 \le u < \infty$. Here, u^* and u_0 are the lowest and greatest values, respectively, of $u(z) = v_Q^{-1}(z)$ in the layer above the conversion discontinuity, the convention being $Q = \{P, S\}$.

Let us write equations (1) and (2) as Stieltjes integrals:

$$\Delta x(p) = p\int_{u_p^*}^{u_p^0} \frac{dH(u)}{(u^2-p^2)^{1/2}} - p\int_{u_s^*}^{u_s^0} \frac{dG(u)}{(u^2-p^2)^{1/2}}, \tag{3}$$

$$\Delta t(p) = \int_{u_p^*}^{u_p^0} \frac{u^2 dH(u)}{(u^2-p^2)^{1/2}} - \int_{u_s^*}^{u_s^0} \frac{u^2 dG(u)}{(u^2-p^2)^{1/2}}, \tag{4}$$

where $0 \le p \le u_Q(z^*+0) = u_Q^* \le u_Q \le u_Q^0$; $Q = \{P, S\}$; $dH(u) \ge 0$, $dG(u) \ge 0$.

The problem we are concerned with here is as follows: use equations (3) and (4) to find $H(u)$ and $G(u)$, hence the depth to the conversion discontinuity, and the lowest and greatest velocities of the relevant waves in the layer above the discontinuity.

Multiply both sides of (3) by $1/(w^2-p^2)^{1/2}$ and (4) by $w^2p/(w^2-p^2)^{1/2}$, and then integrate the resulting expressions over p between p_1 and p_2. The result is

$$f(w) = \int_{p_1}^{p_2} \frac{\Delta x(p)dp}{(w^2-p^2)^{1/2}} = \int_{p_1} tpp_2 \frac{p}{(w^2-p^2)^{1/2}} \left\{ \int_{u_p^*}^{u_p^0} \frac{dH(u)}{(u^2-p^2)^{1/2}} - \int_{u_s^*}^{u_s^0} \frac{dG(u)}{(u^2-p^2)^{1/2}} \right\} dp,$$

$$g(w) = w^2 \int_{p_1}^{p_2} \frac{\Delta t(p)pdp}{(w^2-p^2)^{1/2}} = w^2 \int_{p_1} tpp_2 \frac{p}{(w^2-p^2)^{1/2}} \left\{ \int_{u_p^*}^{u_p^0} \frac{u^2dH(u)}{(u^2-p^2)^{1/2}} - \int_{u_s^*}^{u_s^0} \frac{u^2dG(u)}{(u^2-p^2)^{1/2}} \right\} dp,$$

where $0 \le p_1 \le p \le p_2 \le u_p^* \le w$; $u \le u_s^0$.

Swap the integrals in the right sides of these equations:

$$f(w) = \int_{u_p^*}^{u_p^0} dH(u) \int_{p_1}^{p_2} \frac{pdp}{((w^2-p^2)(u^2-p^2))^{1/2}} - \int_{u_s^*}^{u_s^0} dG(u) \int_{p_1}^{p_2} \frac{pdp}{((w^2-p^2)(u^2-p^2))^{1/2}},$$

$$g(w) = w^2 \int_{u_p^*}^{u_p^0} u^2 dH(u) \int_{p_1}^{p_2} \frac{pdp}{((w^2-p^2)(u^2-p^2))^{1/2}} - w^2 \int_{u_s^*}^{u_s^0} u^2 dG(u) \int_{p_1}^{p_2} \frac{pdp}{((w^2-p^2)(u^2-p^2))^{1/2}}.$$

The inner integrals in the last equations are

$$K(w,u) = \int_{p_1}^{p_2} \frac{pdp}{((w^2-p^2)(u^2-p^2))^{1/2}} = \ln \frac{(w^2-p_1^2)^{1/2} + (u^2-p_1^2)^{1/2}}{(w^2-p_2^2)^{1/2} + (u^2-p_2^2)^{1/2}}.$$

The functions $f(w)$ and $g(w)$ can be written in a form that easily lends itself to numerical integration:

$$f(w) = f_P(w) - f_S(w); \quad g(w) = g_P(w) - g_S(w),$$

where

$$f_Q(w) = x(p_2)\arcsin\frac{p_2}{w} - x(p_1)\arcsin\frac{p_1}{w} - \int_{x_1}^{x_2} \arcsin\frac{t'(x)}{w}dx,$$

$$g_Q(w) = w^2\left\{t(p_1)(w^2 - p_1^2)^{1/2} - t(p_2)(w^2 - p_2^2)^{1/2} + \int_{x_1}^{x_2}(w^2 - t'^2(x))^{1/2}t'(x)dx\right\},$$

$$Q = \{P, S\}.$$

The final forms of the equations from which to determine $H(u)$ and $G(u)$ are

$$f(w) = \int_{u_p^*}^{u_p^0} K(w,u)dH(u) - \int_{u_s^*}^{u_s^0} K(w,u)dG(u), \tag{5}$$

$$g(w) = w^2\left\{\int_{u_p^*}^{u_p^0} u^2K(w,u)dH(u) - \int_{u_s^*}^{u_s^0} u^2K(w,u)dG(u)\right\}. \tag{6}$$

We thus have a set of two Fredholm integral equations in two functions, $H(u)$ and $G(u)$, which are to satisfy the conditions

$$dH(u) \geq 0; \quad dG(u) \geq 0. \tag{7}$$

It is easy to see that $H(u)$ and $G(u)$ as solutions to (5) and (6) will minimize the functionals

$$I(H,\Delta x) = \int_{p_1}^{p_2}\frac{1}{p}\left\{\Delta x(p) - p\int_{u_p^*}^{u_p^0}\frac{dH(u)}{(u^2-p^2)^{1/2}} + p\int_{u_s^*}^{u_s^0}\frac{dG(u)}{(u^2-p^2)^{1/2}}\right\}^2 dp, \tag{8}$$

$$J(H,\Delta t) = \int_{p_1}^{p_2} p\left\{\Delta t(p) - \int_{u_p^*}^{u_p^0}\frac{u^2dH(u)}{(u^2-p^2)^{1/2}} + \int_{u_s^*}^{u_s^0}\frac{u^2dG(u)}{(u^2-p^2)^{1/2}}\right\}^2 dp. \tag{9}$$

This means that least squares solutions will be considered as the solutions to (5) and (6).

The problem of determining $u_p(z) = v_p^{-1}(u)$ and $u_s(z) = v_s^{-1}(u)$ from differences between travel times of converted shear and direct compressional waves thus reduces to a problem in quadratic programming on an infinite-dimensional space, minimizing the quadratic functionals (8) and (9) under the linear restrictions (7).

The ray parameter p takes on values in $[p_1, p_2]$, where $p_1 = t'(x_{\min}) \geq 0$ and $p_2 = t'(x_{\max}) \leq u^*$, when the travel time curves are convex downward; $p_1 = t'(x_{\max}) \geq 0$ and $p_2 = t'(x_{\min}) \leq u^*$, when they are convex upward. The above range for the ray parameter should be wide enough. In that case the determination of seismic wave velocities will be more stable. For this requirement to be satisfied, converted earthquake waves are to be recorded in a wide range of either depth or epicentral distance.

Knowledge of derivatives of the P and SV travel times is required for calculating the integrals on the left-hand sides of (5) and (6) and the kernel K, the derivatives being equal to the respective ray parameters at the extreme points only of the travel time curves.

NUMERICAL SOLUTION

1. We proceed to solve (5) and (6) under the condition (7) by passing to a discrete formulation, i.e., dividing $[u_p^*, u_p^0]$ and $[u_s^*, u_s^0]$ by N points into $N-1$ subsegments $[u_{j-1}, u_j^p]$ and $[u_{j-1}^s, u_j^s]$. Consider the jump functions ΔH_j and ΔG_j (j = 1,2,..., N) on the segments $[u_p^*, u_p^0]$ and $[u_s^*, u_s^0]$. In that case we shall have two sets of $2N$ equations that are linear in $2N$ unknown ΔH_j and ΔG_j and nonlinear in the four unknowns u_p^*, u_p^0, u_s^*, and u_s^0, which can be written in the form

$$Ay - f = 0; \quad By - g = 0; \tag{10}$$

$$\Delta H_j \geq 0, \quad \Delta G_j \geq 0; \tag{11}$$

where

$$A = \{a_{ij}\}; \quad B = \{b_{ij}\}; \quad f^T = \{f_i\}; \quad g^T = \{g_i\};$$

$$y^T = \{\Delta H_j\}; \quad i = 1,2,\ldots,N; \quad j = 1,2,\ldots,N;$$

$$a_{ij} = \ln \frac{(w_i^2 - p_1^2)^{1/2} + (u_j^2 - p_1^2)^{1/2}}{(w_i^2 - p_2^2)^{1/2} + (u_j^2 - p_2^2)^{1/2}},$$

$$u_j \in [u_p^*, u_p^0]; \; w_i \in [u_p^*, u_p^0] \quad \text{when} \quad \begin{cases} i = 1,2,\ldots,N \\ j = 1,2,\ldots,N, \end{cases}$$

$$u_j \in [u_p^*, u_p^0]; \; w_i \in [u_s^*, u_s^0] \quad \text{when} \quad \begin{cases} i = N+1, N+2,\ldots,N \\ j = 1,2,\ldots,N \end{cases},$$

$$a_{ij} = -\ln \frac{(w_i^2 - p_1^2)^{1/2} + (u_j^2 - p_1^2)^{1/2}}{(w_i^2 - p_2^2)^{1/2} + (u_j^2 - p_2^2)^{1/2}},$$

$$u_j \in [u_s^*, u_s^0]; \; w_i \in [u_p^*, u_p^0] \quad \text{when} \quad \begin{cases} i = 1,2,\ldots,N \\ j = N+1, N+2,\ldots,2N \end{cases}$$

$$u_j \in [u_s^*, u_s^0]; \; w_i \in [u_s^*, u_s^0] \quad \text{when} \quad \begin{cases} i = N+1, N+2,\ldots,2N \\ j = N+1, N+2,\ldots,2N \end{cases}$$

$$b_{ij} = w_i^2 u_j^2 \ln \frac{(w_i^2 - p_1^2)^{1/2} + (u_j^2 - p_1^2)^{1/2}}{(w_i^2 - p_2^2)^{1/2} + (u_j^2 - p_2^2)^{1/2}},$$

$$u_j \in [u_p^*, u_p^0];\ w_i \in [u_p^*, u_p^0] \quad \text{when} \quad \begin{cases} i = 1,2,\ldots,N \\ j = 1,2,\ldots,N \end{cases}$$

$$u_j \in [u_p^*, u_p^0];\ w_i \in [u_s^*, u_s^0] \quad \text{when} \quad \begin{cases} i = N+1, N+2,\ldots,2N \\ j = 1,2,\ldots,N \end{cases},$$

$$b_{ij} = -w_i^2 u_j^2 \ln \frac{(w_i^2 - p_1^2)^{1/2} + (u_j^2 - p_1^2)^{1/2}}{(w_i^2 - p_2^2)^{1/2} + (u_j^2 - p_2^2)^{1/2}},$$

$$u_j \in [u_s^*, u_s^0];\ w_i \in [u_p^*, u_p^0] \quad \text{when} \quad \begin{cases} i = 1,2,\ldots,N \\ j = N+1, N+2,\ldots,2N \end{cases}$$

$$u_j \in [u_s^*, u_s^0];\ w_i \in [u_s^*, u_s^0] \quad \text{when} \quad \begin{cases} i = N+1, N+2,\ldots,2N \\ j = N+1, N+2,\ldots,2N \end{cases}.$$

Suppose u_p^*, u_p^0 and u_s^*, u_s^0 are given, i.e., are known. Solution of (10) will yield the distribution of P and SV velocities, hence the function $k(z) = v_p(z)/v_s(z)$.

Solution of sets of linear equations (10) under the linear restrictions (11) and for given values of u_p^*, u_p^0 and u_s^*, u_s^0 is a problem in quadratic programming on a finite-dimensional space [7]. When the matrices A and B are nonsingular, then this quadratic programming problem has a unique solution, because the functionals I and J (convex in y) are bounded from below and continuous on the convex set $Y = \{y \mid y \geq 0\}$ [7].

The functions $H(u)$ and $G(u)$ in $[u_p^*, u_p^0]$ and $[u_s^*, u_s^0]$ can be found from the relations

$$H(u) = H_n = \sum_{j=1}^{n} \Delta H_j;\quad u \in [u_j^p, u_{j+1}^p];$$

$$\Delta H_j \geq 0,\quad n = 1,2,\ldots,N;\quad H_0 = H(u_p^*) = 0,$$

$$G(u) = G_n = \sum_{j=1}^{n} \Delta G_j;\quad u \in [u_j^s, u_{j+1}^s],$$

$$\Delta G_j \geq 0,\quad n = 1,2,\ldots,N;\quad G_0 = G(u_s^*) = 0,\quad H_N = G_N.$$

2. The standard procedure to deal with problems in quadratic programming is ti use a conjugate gradient method. This method always converges in a finite number of steps, but requires an initial, or zero, approximation [8]. This may be the mean seismic velocity in the layer and the corresponding thickness.

Consider the problem of determining seismic velocity in a layer and the position of the conversion discontinuity under the assumption that the P and SV velocities in the layer are constant. In that case $n = 1$; $u_i = u$; $w_j = w$; u, w = const; $H(u) = G(w) = \Delta z$ and equations (5), (6) will be written as

$$F(u) = \{K(u) - K(u,w)\}\Delta z;$$

$$f(w) = \{K(u,w) - K(w)\}\Delta z;$$

$$g(u) = \{w^2 K(u,w) - u^2 K(u)\}\Delta z;$$

$$g(w) = \{w^2 K(w) - u^2 K(u,w)\}\Delta z.$$

We thus have four equations in three unknowns. It can however be shown that these equations are not independent, one of them being a combination of the other three.

Numerical experiments showed that the best way to solve these equations was as follows. The first two equations give

$$\Delta z = 0.5 \left[\frac{f(w)}{K(u,w) - K(w)} - \frac{f(u)}{K(u) - K(u,w)}\right].$$

Substituting Δz into the other two equations, we get

$$w^2 K(u,w) - u^2 K(u) = G_1;$$

$$w^2 K(w) - u^2 K(u,w) = G_2,$$

where $G_1 = g(u)/\Delta z$; $G_2 = g(w)/\Delta z$.

We are going to solve these equations by simple iteration. To do this, we write u^2 and w^2 in the forms

$$u^2 = [G_1 K(w) - G_2 K(u,w)]/\det;$$

$$w^2 = [G_1 K(u,w) - G_2 K(u)]/\det,$$

where $\det = K^2(u, w) - K(u)K(w)$.

Solving the last two equations will yield u and w. Substituting these into the equation, we find Δz. The method of simple iteration converges to the solution in the case we are considering for practically any initial approximation, i.e., is stable enough.

The end values of $[u_p^*, u_p^0]$ and $[u_s^*, u_s^0]$ should be specified to solve the problem. These are usually not known. The method we use here is as follows. Since the desired distributions of P and SV velocities in a layer are functions of the coordinate z: $u_Q = u_Q(z)$ and $u_Q(0) = u_Q^0$, $u_Q(z^*) = u_Q^*(Q = \{P, S\})$, we will extend the functions $u_Q = u_Q(z)$ so as to make the new functions $\tilde{u}_Q = \tilde{u}_Q(z)$ to become identical with $u_Q = u_Q(z)$ everywhere on the segment $[0, z^*]$, except perhaps at a finite number of points on the segment. To be specific, we shall preserve monotonicity for $\tilde{u}_Q = \tilde{u}_Q(z)$ by extending $u_Q = u_Q(z)$ at the ends of the segment $[0, u^*]$ setting $u_0 = u_{\max} \geq u_Q(0)$ and $u^* = u_{\min} = p_2 \leq u_Q(z^*)$. Here, $u_{\max}$ corresponds to the maximum value of the index of refraction $u_Q(z)$ at the top of the $0z^*$ layer and can usually be specified *a priori*; $u_{\min}$ is not below the derivative of P travel times at the rightmost point of it or, e.g., when the crust is studied, not below the reciprocal of mantle velocity just below the Moho. In that case the integrals (3) and (4) will not be affected and, when the problem has a unique solution, the resulting distribution $\tilde{u}(z)$ will be different from the true distribution in layers of zero thickness only. The end values of u_{max} and u_{min} are revised by iteration during the numerical solution of the problem. This is done as follows. At the first step the problem is solved for the end values $u^{*0} = u_{min}$ and $u_0^{\,0} = u_{max}$. The next u^* and u_0 will be those end values u_i and u_j for which ΔH_i, ΔH_j and ΔG_i, ΔG_j do not vanish (are above zero to within a specified error). The problem is again solved, but this time for the new end values $u^* = u_i$ and $u_0 = u_j$. The procedure is repeated, until the end values of ΔH_1, ΔH_N and ΔG_1, ΔG_N are different from zero.

Note that the left and right sides of (5) and (6) contain the ray parameter for direct compressional wave, which is numerically equal to the derivative of $t(x)$. The experimental travel time curve is known to within some uncertainty as a discrete set of points; in that case it should be fitted with a curve which is convex upward or downward, depending on which branch of the travel times of the refracted wave (before or after the inflexion point, direct or reversed branch) is under consideration. A convex cubic spline $T(x)$ should be taken to do the job [4].

In order to find the integrals on the left-side parts of (5) and (6) we should determine the points in the travel time curves of *P* and *SV* which have identical values of the ray parameters p_1 and p_2. This can be done by using interpolation relations for the cubic spline itself and its first derivative in terms of its second derivative at grid points [1].

Now we turn to examine the results of this mathematical modeling. Our example will be concerned with the determination of *P* and *SV* velocities in the crust for a model close to the standard. The theoretical travel times of *P* and *SV* when reduced to 9 km/s are shown in Fig. 1. The curves are for seismic waves going upward from a seismic source situated at a depth of 250 km, the conversion taking place at the Moho, at a depth of 43 km. The derivatives of travel times at the ends equal the reciprocals of apparent velocities (about 12 and 10 km/s, respectively). The compressional velocities in the crust as derived from the differences of *P* and *SV* travel times are shown in Fig. 2. The original velocity curve represents a two-layered crust with *P*-wave velocity being 6 km/s in the upper layer and 6.75 at the top and 6.84 km/s at the bottom of the lower layer. The *S*-wave velocities were derived from *P*-wave velocities by dividing them by 1.78. The end values of the velocity were 4 and 8 km/s for *P* wave and 2.3 and 4.6 km/s for *S* waves. Each velocity interval was divided into 49 subsegments. The integrals in the left sides of (5) and (6) were found using the Simpson formula.

Figure 2, *a* presents velocity curves reconstructed on the assumption that compressional and shear velocities were unknown throughout the crust. One can see that the mean thickness and velocity in a layer were determined well enough, but the velocity curve does not in the least reflect the velocity jumps within the layers.

Figure 2, *b* presents velocity curves reconstructed on the assumption that compressional velocity was known in the upper layer but unknown in the lower. In that case the resulting velocity distribution in the layer is close to that given in the model.

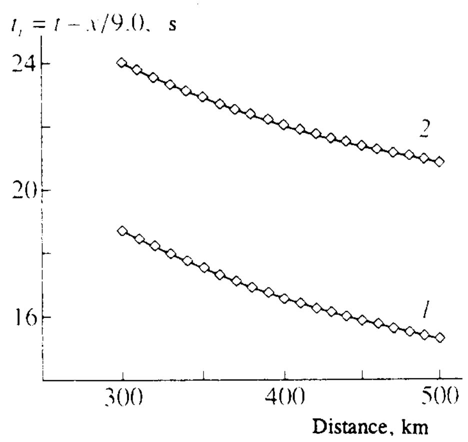

Figure 1 Travel times of direct compressional (*1*) and converted shear (*2*) wave.

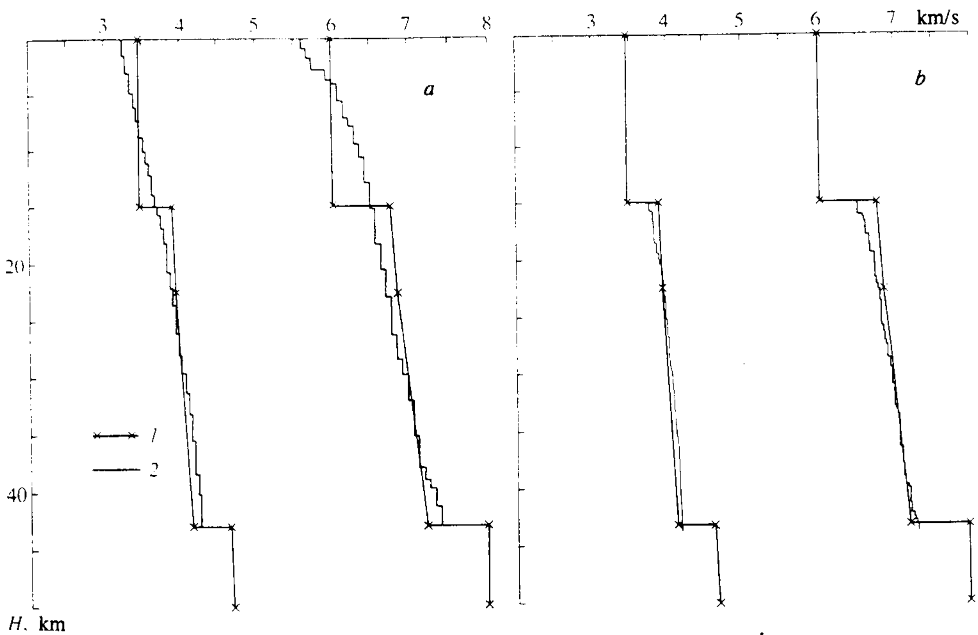

Figure 2 Reconstructed velocity distributions for a two-layer model (*a*) and one layer from a two-layer model (*b*) in the crust from the travel times shown in Fig. 1: *1* – original velocity distributions, *2* – reconstructed distributions.

It can be concluded from these results that, when treated in the class of monotone positive step-functions, the problem of determining compressional and shear velocities from travel time differences of converted shear and direct compressional waves has a unique solution, because it does not depend on the choice of the end values, u_{max} and u_{min}. It should however be pointed out that the determination of compressional velocity alone with a fixed ratio between compressional and shear velocity is more stable than the simultaneous determination of compressional and shear velocities.

CONCLUSION

It is shown that compressional and shear velocities can be determined simultaneously from travel times of converted shear and direct compressional seismic waves in a vertically varying earth.

Travel times of P and SV waves can be obtained by observing earthquakes with identical hypocentral depths, but at different epicentral distances at a single recording site. The greater the range of the ray parameters, the more stable is the determination of the velocity distribution in a layer of interest. Observation of this kind would give a travel time curve for the vertically varying velocity structure beneath the recording site only (under the seismograph station). When a sufficient number of recording sites are available, a 3-D distribution of seismic velocity in a layer of interest can be derived.

REFERENCES

1. J. H. Ahlberg, E. N. Nilson, and J. L. Walsh, *The Theory of Splines and Their Applications* (New York - London: Academic Press, 1967).
2. S. S. Andreev, *Izv. AN SSSR. Ser. Geofiz.* N1: 21–29 (1957).
3. N. K. Bulin, *Izv. AN SSSR. Ser. Geofiz.* N6: 781–786 (1960).
4. V. Yu. Burmin, *Izv. AN SSSR. Fizika Zemli* N2: 90–96 (1980).
5. E. I. Galperin, R. M. Galperina, A. V. Frolova, and M. S. Erenburg, *Wave Fields in the Method of Converted Earthquake Waves* (in Russian) (Moscow: OIFZ RAN, 1995).
6. M. L. Gerver and V. M. Markushevich, *Computational Seismology* (in Russian), iss. 3 (Moscow: Nauka, 1967): 3–51.
7. V. G. Karmanov, *Mathematical Programming* (in Russian) (Moscow: Nauka, 1975).
8. M. M. Nikashova, *An Algorithm for Quadratic Programming by the Conjugate Gradient Method* (in Russian) (Moscow: VTs MGU, 1968).
9. I. V. Pomerantseva and A. N. Mozzhenko, *Seismic Investigations with the "Zemlya" System* (in Russian) (Moscow: Nedra, 1977).
10. M. Hasegawa, *Z. Geophysik* **6**, N2: 78–98 (1930).