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Seismic Velocities in the Mantle and the Radius of the Core as Deduced from *P* and *PcP* Travel Times

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Refraction and reflection travel times curve were inverted to determine seismic velocity distribution in the mantle and the depth to the outer core using data from earthquakes and nuclear explosions reported in the literature.

INTRODUCTION

In 1939 Harold Jeffreys [25] determined the Earth's velocity structure using seismic travel times from earthquakes and found a value of 3473±3 km for the radius of the core [24] using the times of *S* and *ScS*. The *PcP* travel times were not used because of a large scatter.

Gutenberg and Richter [21] used the *PcP* times and obtained a similar value.

There have been many attempts to improve the mantle velocity distribution and the radius of the core using travel times from high-yield nuclear explosions. Most of the methods used to determine the depth to the core were based on fitting the depth and seismic velocity above the core boundary to the times of *PcP* derived from the mantle velocity distribution based on *P* times.

Taggart and Engdahl [27] used new tables of *P* times for earthquakes and nuclear explosions [23] and got a value of 3477±2 km for the radius of the core.

Kogan [9] thought the most likely range of value to be 3477-3485 km.

Yanovskaya [12] used Taggart and Engdahl's data, converted the reflection times of Herrin *et al.* [23] to reflection times [11], and found a 90-percent confidence interval for the radius of the core and for the velocity gradient above the core boundary. The end values for the radius in this interval were 3474.5 and 3478.8 km.

Bolt [14] proposed a technique for determining the core radius and the velocity at the base of the mantle from apparent velocities of grazing rays and the times of *PcP* along vertical rays. His determination yielded 3475±2 km for the radius [15].

Later studies yielded 10-15 km larger values of the radius compared to the Jeffreys value. The mantle model used was a standard model based on oscillation data [17].

Gilbert *et al.* [20] supposed the radius of the core to be 3482-3485 km.

Hales and Roberts [22] used differences between the times of *ScS* and *S* and two different shear velocity distributions for the lower mantle and found the values of 3489.92 ± 4.66 and 3486.10 ± 4.59 km for the core radius.

Engdahl and Johnson [18] inverted the travel times of short period *PcP* and *P* by the Backus-Gilbert method and obtained 3484.2 ± 2.9 km.

Jordan [26] used time differences between short period *PcP* and *P* due to shallow events and long period *PcP* and *P* from deep events and found a value of 3485 km.

At present the accepted value of the core radius is 3480 km [16]. It was derived by a joint interpretation of body-wave and oscillation data and so is a better fit to both kinds of data. The large scatter of the radius values based on body wave times is obviously due to two causes: one of them consists in drawbacks of the travel time interpretation, the other, in the heterogeneity of data used for interpretation. We examined all data available on body-wave travel times and interpreted them using a technique described below. We examined Kogan's [8], [9] smoothed times of *P* and *PcP* and raw times of *P* obtained by Herrin *et al.* (these data can be found in [13]) and of *PcP* found by Taggart and Engdahl [27].

Determination of Velocity Distribution from the Times of Refracted Waves. Velocity distribution in the mantle was determined on the assumption that the Earth's surface is a plane. It was therefore necessary to convert the times obtained in the θ, t coordinates to those in x, t [5]. Here, θ is the distance in degrees for a spherical Earth; x is the distance in kilometers for a plane Earth; t is the travel time, the same for both earth models. The coordinates θ and x are related by

$$x = \pi \theta R / 180,$$

where R is the Earth's radius.

The result of solving the plane problem is a velocity distribution $v(z)$ where the depth z is measured from the ground surface. The conversion from $v(z)$ to $v(r)$, where r is the distance from the Earth's center to the point considered, was accomplished as follows [5]

$$r = R\exp[-z/R], \quad v(r) = \exp[-z/R]\, v(z).$$

We will henceforth distinguish two types of seismic waves propagating in an elastic medium.

1. Refracted (transmitted) waves corresponding to ray paths with different depths of maximum penetration z_m. One has the relation $z(\psi) < z(\varphi)$ for any two rays having the angles of emergence φ, $\psi \in [0, \pi/2]$ measured from the vertical.

2. Simply reflected waves associated with velocity discontinuities at the depths $z = z_i^*$ ($i = 1, 2, \ldots, n$). One has the relation $z(\varphi) = z^*$ for a wave reflected at the discontinuity $z = z^*$, for all rays having the angles of emergence $\varphi \in [0, \varphi_c]$, where φ_c is the angle of a critical ray.

The travel time equations can be written in the parametric form

$$x(p) = 2p\int_0^{z_m} \frac{dz}{(v^{-2}(z) - p^2)^{1/2}},$$
$$t(p) = 2p\int_0^{z_m} \frac{dz}{v^2(z)(v^{-2}(z) - p^2)^{1/2}}, \tag{1}$$

where $p=t'(x)$ is the ray parameter numerically equal to the derivative of the travel time curve at the respective point; z_m is the depth of maximum penetration for a ray with parameter p.

Since a seismic ray starts at the ground surface, its turning point is (x_m, z_m). One has $z_m=z(p)$ for a refracted wave at this point and $z_m=z^*$ for a reflected wave.

If the velocity function $v=v(z)$ is known to some depth $z=z^*$, then the inversion of refracted-wave times can be written as

$$z(q) = z^* + \frac{1}{\pi}\int_0^{x(q,u^*)} \cosh^{-1}(p/q)\,dx(p,u^*), \tag{2}$$

where

$$x(p,u^*) = x(p) - 2p\int_0^{z^*} \frac{d\eta}{(v^{-2}(\eta) - p^2)^{1/2}};$$
$$z^* \leq z_m \leq z_M;\quad q = t'(x_m).$$

Formula (2) yields the depth of maximum penetration for a ray emerging at distance $x(p)$ from the source. Combined with

$$v^{-1}(z_m) = u(z_m) = t'[x(q)], \tag{3}$$

which gives seismic velocity at depth $z_m=z(q)$, it provides the desired relation $v=v(z)$ in a parametric form for $z^*\leq z_m\leq z_M$; $x(p, u^*)$ is the distance from the point where the ray penetrating to depth $z_m\geq z^*$ went down across $z=z^*$ to the point where it came back to depth $z=z^*$; it is a function of the ray parameter p.

If $z^*=0$ and the travel time function is a simple smooth curve with a steadily decreasing derivative, then [10]

$$z(q) = \frac{1}{\pi}\int_0^{x(q)} \cosh^{-1}(p/q)\,dx(p,u^*). \tag{4}$$

In those cases where the travel time curve has cusps, one has to divide the curve into a forward and a backward segment before using (2) and interpret the segments successively.

If there is a layer of constant velocity (which may be a low velocity layer) starting at depth z^*, then the inversion is given by (2) combined with the formulas determining the seismic velocity in the layer and its thickness [2]:

$$v^* = (v(\bar{z}^*)v)^{1/2},\quad \Delta z^* = \frac{\Delta x}{2}\left[\frac{v(\bar{z}^*)}{v} - 1\right]^{1/2}, \tag{5}$$
$$\bar{z}^* = z^* + \Delta z^*,$$

where $v=\Delta x/\Delta t$. The quantities $v(\overline{z^*})$, Δx, Δt are determined directly from the travel times after $v(z)$ has been found in the layer Oz^* using the first segment. It is important that the parameters of this layer are determined from (5) without using the times of the reflected and the head wave.

Suppose the observed refraction travel time curve consists of a finite number of forward and backward segments [2] and is given as a discrete set of points with some random errors. In order to be able to use the inversion formulas, one has to fit separate segments of the travel time curve by some functions $T(x)$ that are convex upward for the forward segments and downward for the backward ones. Earlier [1], these functions were constructed as convex splines that minimize the functional

$$S = \sum_{i=1}^{N} [t(x_i) - T(x_i)]^2. \tag{6}$$

The construction of a convex spline reduces to a problem of quadratic programming [6], [7] with respect to the second derivatives $T''(x)$ of $T(x)$ at the observation sites x_i ($i=1, 2,\ldots, N$) [1].

Once $T(x)$, and hence the ray parameters $p=T'(x)$ at each point of the segment $[0, x_M]$, have been determined, velocity variation with depth can be found using

$$v^{-1}(z_m) = T'(x_m)$$

and (2) with a suitable quadrature formula [3], [4].

The medium is assumed to consist of plane parallel layers. The layer boundaries are chosen in such a manner that each point of the observed travel time function is the critical point of the refracted wave in the respective layer (first point is on the ground surface). In this way the medium is divided into layers corresponding to the data points, the number of layers being one less than that of the data points. Since the observed travel time curve is fitted by the following cubic parabola in each segment $[x_i, x_{i+1}]$

$$T(x) = a_{i0} + a_{i1}(x - x_i) + a_{i2}(x - x_i)^2 + a_{i3}(x - x_i)^3,$$

where the constants a_{ij} ($j=0, 1, 2, 3$) are expressible in terms of the spline and its second derivatives at the points of the grid Δ:

$$0 = x_0 < x_1 < \ldots < x_N = x_m,$$

$$a_{i0} = \frac{1}{6h_i}\{-x_i^3 T''_{i+1} + x_i h_i^2 T''_{i+1} + x_{i+1}^3 T''_i - x_{i+1} h_i^2 T_i + 6(-x_i T_{i+1} + x_{i+1} T_i)\},$$

$$a_i = \frac{1}{6h_i}\{3x_i^2 T''_{i+1} - h_i^2 T''_{i+1} - 3x_{i+1}^2 T''_i - h_i^2 T''_i + 6(T_{i+1} - T_i)\},$$

$$a_{i2} = \frac{x_{i+1} T''_i - x_i T''_{i+1}}{2h_i}, \quad a_{i3} = \frac{T''_{i+1} - T''_i}{6h_i},$$

the inversion formula (2) can be written as

$$z_N = z_m = \frac{1}{\pi}\sum_{i=1}^{N-1}\int_{x_i}^{x_{i+1}} \cosh^{-1}\left[\frac{a_{i1} + 2a_{i2}(x - x_i) + 3a_{i3}(x - x_i)^2}{T'_N}\right] dx.$$

The last formula, which is an exact one for a cubic spline, can be written in the form

$$z_N = z_{N-1} + \frac{1}{\pi}\int_{x_i}^{x_{i+1}} \cosh^{-1}\left[\frac{b_1 + 2b_2^* + 3b_3 x^{*2}}{T_N'}\right] dx, \tag{7}$$

where

$$x^* = x - 2p\int_0^{z_{N-1}} \frac{dz}{(u^2(z)-p^2)^{1/2}}; \quad 0 \le x^* \le x_{N-1}^*;$$

$$T_N' \le p = b_1 + 2b_1 x^* + 3b_3 x^{*2} \le T_{N-1}'.$$

The constants b_j ($j=1, 2, 3$) are found from the linear equations

$$\begin{cases} b_1 = T_{N-1}', \\ b_2 x_N^{*2} + b_3 x_N^{*3} = T(x_N^*) - T_{N-1}' x_N^*, \\ 2b_2 x_N^* + 3b_3 x_N^{*2} = T_N' - T_{N-1}', \end{cases}$$

$$x_N^* = x_N - 2p\int_0^{z_{N-1}} \frac{dz}{(u^2(z)-p^2)^{1/2}}; \quad p = T_N'; \tag{8}$$

$$T(x_N^*) = T(x_N) - 2\int_0^{z_{N-1}} \frac{u^2(z)dz}{(u^2(z)-p^2)^{1/2}}; \quad T_{N-1}' = a_{N1};$$

$$T_N' = a_{N1} + 2a_{N2}(x_{N+1} - x_N) + 3a_{N3}(x_{N+1} - x_N)^2.$$

The solution of (8) is

$$b_2 = \frac{1}{x_N^*}\{3T(x_N^*)/x_N^* - T_{N-1}' - T_N'\},$$

$$b_3 = -\frac{1}{x_N^{*2}}\{2T'(x_N^*)/x_n^* - T_{N-1}' - T_N'\}.$$

To find the integral in (7), divide the segment $[0, x_N^*]$ into n subsegments at equal steps of $h=x_N^*/n$ and approximate the quadratic parabola on each subsegment by the linear function

$$\varphi_j(x^*) = c_{1j} + c_{2j}(x^* - x_j^*), \quad j = 1, 2, \ldots, n+1.$$

The result is

$$z_N = z_{N-1} + \frac{1}{\pi}\sum_{j=1}^{n}\int_{x_j^*}^{x_{j+1}^*} \cosh^{-1}\left[\frac{c_{1j} + c_{2j}(x^* - x_j^*)}{T_N'}\right] dx^* =$$

$$= z_{N-1} + \frac{T_n}{\pi}\sum_{j=1}^{n}\frac{1}{c_{2j}}\left\{y_j\cosh^{-1}(y_i) - (y_j^2-1)^{1/2} - \frac{c_{1j}}{T_N'}\cosh^{-1}\left(\frac{c_{1j}}{T_N'}\right) + \left[\left(\frac{c_{1j}}{T_N'}\right)^2 - 1\right]^{1/2}\right\},$$

where

$$y = \frac{c_{1j} + c_{2j}(x_{j+1}^* - c_j^*)}{T_N} = \frac{c_{1j} + c_{2j}h}{T_N}.$$

The constants c_{1j} and c_{2j} can be expressed in terms of the coefficients of the cubic parabola as follows:

$$c_{1j} = b_1 + 2b_2 x_j^* + 3b_3 x_j^{*2},$$
$$c_{2j} = 2b_2 + 3b_3(x_{j+1}^* + x_j^*).$$

Inversion of the Times of Refracted Waves. Suppose the velocity function has the discontinuity $\Delta v = v(z^* - 0) - v(z^* + 0)$ at some depth $z=z^*$. In that case the times of the refracted wave will have a discontinuity. If the segment of the travel time curve corresponding to the wave propagating in the layer above the discontinuity $z=z^*$ is completely specified, then the position of that discontinuity can be exactly determined from the corresponding segment [2]. However, for various reasons it is not possible to measure refracted times fully, consequently, the depth to the discontinuity will involve some error. In particular, when trying to determine the depth to the core, we find that the times of a refracted wave continuously become those of a diffracted wave, which makes it hard to find the critical point of the refracted-wave times. Determination of the core boundary and of the velocity above that boundary requires the use of a reflection travel time curve.

Consider the travel time equations for a wave reflected at a discontinuity at depth $z=z^*$ and try to locate the $z=z^*$ discontinuity and fine the distribution of seismic velocity in the Oz^* layer. Gerver and Markushevich [5] showed that $v(z)$ cannot be determined from travel times uniquely, because the same curve $t(x)$, $x \in [0, x^*]$ corresponds to different velocity functions $v(z)$ having the same measure $H(u)$ where

$$H(u) = mes\{z: \ z \leq z^*, \ \ v^{-1}(z) \leq u\}.$$

We are to find $H(u)$ and, hence the depth to the reflector and the maximum and minimum velocities in the layer above it. By definition, $H(u)$ is (a) nondecreasing, (b) vanishes for $-\infty < u \leq u^*(z^* - 0)$, and ($c$) equals h when $u(0) = u_0 \leq u < \infty$. Here, u^* and u_0 are the minimum and maximum values, respectively, of $u(z) = v^{-1}(z)$ in the layer above the reflector.

Write the first equation of (1) as a Stieltjes integral:

$$x(p) = 2p \int_{u^*}^{u_0} dH(u)/(u^2 - p^2)^{1/2}, \tag{9}$$

where $0 \leq p \leq u^* \leq u \leq u_0$; $dH(u) \geq 0$, that is, $H(u)$ is a monotonic function.

Equation (9) is a Fredholm-type integral equation of the first kind. The problem of solving it, even for a monotonic $H(u)$, is ill posed, because its left-hand side may, in the general case, not belong to the region of the integral operator [17]. For this reason we will solve (9) by least squares minimizing the functional

$$J(H,x) = \int_0^{u^*} \frac{0.5}{p} \left\{ x(p) - 2p \int_{u^*}^{u_0} \frac{dH(u)}{(u^2-p^2)^{1/2}} \right\}^2 dp \tag{10}$$

given that

$$dH(u) \geq 0. \tag{11}$$

The problem of determining $u(z) = v^{-1}(z)$ using a reflection travel time curve is thereby reduced to quadratic programming in an infinite-dimensional space: minimization of (10) under the linear constraints (11).

A necessary condition for the minimum of $J(H, x)$ is that its first variation should vanish [19]:

$$\delta J = \frac{d}{d\alpha} J[H(u) + \alpha \delta H, x]_{\alpha=0}.$$

After some manipulations we find that the element that minimizes $J(H, x)$ is the solution to

$$f(w) = \int_0^{u^*} \frac{x(p)dp}{(w^2-p^2)^{1/2}} = \int_{u^*}^{u_0} dH(u) \int_0^{u^*} \frac{2p\,dp}{((w^2-p^2)(u^2-p^2))^{1/2}} du, \tag{12}$$

where $u^* \leq w \leq u_0$.

The inner integral on the right-hand side of (12) is

$$K(u, w) = \int_0^{u^*} \frac{2p\,dp}{((w^2-p^2)(u^2-p^2))^{1/2}} = 2 \ln \frac{w+u}{(w^2-u^{*2})^{1/2} + (u^2-u^{*2})^{1/2}}.$$

Equation (12) can be written in a different form. Changing of variables $\xi = \sin^{-1}(p/w)$ in the left-hand side of (12) and integrating the result by parts, we get

$$f(w) = \int_0^{u^*} \frac{x(p)dp}{(w^2-p^2)^{1/2}} = \int_0^{\sin^{-1} u^*/w} x(w, \xi)\, d\xi =$$

$$= x(u^*) \sin^{-1} \frac{u^*}{w} - \int_0^{x(u^*)} \sin^{-1} \frac{t'(x)}{w} dx.$$

The ultimate result is a Fredholm-type integral equation of the first kind with a symmetrical positive kernel:

$$f(w) = x(u^*) \sin^{-1} \frac{u^*}{w} - \int_0^{x(u^*)} \sin^{-1} \frac{t'(x)}{w} dx =$$

$$= 2 \int_{u^*}^{u_0} \ln \frac{w+u}{(w^2-u^{*2})^{1/2} + (u^2-u^{*2})^{1/2}} dH(u)$$

the solution of which is sought in the set $\Re(u)$ of monotonic functions $H(u) \in \Re(u)$.

The functional (10) and equation (12) fit a complete reflection travel time curve. The outer integration limits 0 and u^* in (10) are equal to its derivatives at the origin ($x=0$) and at the critical point ($x=x^*$) of the curve $t=t(x)$. However, because subcritical reflections have low amplitudes, and there is a complex interference pattern around the critical point or the waves are entirely outside the observation interval, travel times can be observed in the interval $[x_{min}, x_{max}]$ which is narrower than ($0 \le x_{min}, x_{max} \le x^*$). As a result, the limits of outer integration in (10) are $p_1 = t'(x_{min})$ and $p_2 = t'(x_{max})$.

The limits of the inner integral in (10), u^* and u_0, are generally not known either. It is known that $u_{max} \ge u_0$ and $u_{min} = p_2 \le u^*$ are: u_{max} corresponds to the greatest index of refraction $u(z)$ at the upper boundary of the Oz^* layer and can be assigned beforehand, $u_{min} = \hat{p}_2$ being equal to the derivative of the travel time curve at its rightmost point.

The question is how can equation (12) be solved under the constraint (11)? Suppose the desired distribution is a function $H(u)$. As has been mentioned, $H(u)=0$ for $\infty < u \le u^*$ and $u \le u_0 < \infty$. Therefore, if one extends the segment $[u^*, u_0]$ to $[u_{min}, u_{max}]$, then obviously, the value of (9) will not change and, because the solution is unique, the resulting distribution $\tilde{H}(u)$ may differ from the true distribution only in layers of zero thickness. The values of u_0 and u^* can thus be assigned in an arbitrary manner on the intervals $[u(0), u_{max}]$ and $[u_{min}, u(z^*)]$.

When one has an incomplete travel time curve, (10) and (12) become, respectively,

$$J(H,x) = \int_{p_1}^{p_2} \frac{0.5}{p} \left\{ x(p) - 2p \int_{p_2}^{u_{max}} \frac{dH(u)}{(u^2-p^2)^{1/2}} \right\}^2 dp, \tag{10a}$$

$$f(w) = 2 \int_{p_2}^{u_{max}} \ln \frac{(w^2-p_1^2)^{1/2} + (u^2-p_1^2)^{1/2}}{(w^2-p_2^2)^{1/2} + (u^2-p_2^2)^{1/2}} \, dH(u), \tag{12a}$$

where

$$f(w) = x_{max} \sin^{-1} \frac{p_2}{w} - x_{min} \sin^{-1} \frac{p_1}{w} - \int_{x}^{x_{max}} \sin^{-1} \frac{t'(x)}{w} \, dx.$$

Consider the function $\psi(u) = H'(u) \ge 0$, $u \in [\overline{u}^*, u^*]$, that is, a positive function $\psi(u)$ in the layer above the reflector with

$$\psi(u) = -dz(u)/du \ge 0,$$

that is, one deals with a monotonically increasing velocity function $v(z)$.

Substituting $\psi(u)$ in (10), one gets a new equation similar to (10). Solving it in the same manner as in the case of (10), one finds $\psi(u)$ and the thickness of the layer above the reflector. Thus, the velocity function above the source can be reconstructed in the class of monotonic functions using any fragment of the travel time curve. The longer the fragment, the more certain is the result.

The above inversion of refracted-wave times was based on the assumption that the layer of unknown velocity starts at the ground surface. Obviously, this problem can be attacked in exactly the same manner assuming that velocity distribution is known to the depth $z=z_0$, for instance, from refraction times. In that case one can replace $x(p)$ with the function

$$x_1(p) = x(p) - 2p\int_0^{z_0} \frac{dz}{(u^2(z) - p^2)^{1/2}} = 2p\int \frac{dH(u)}{(u^2 - p^2)^{1/2}}.$$

Now consider the numerical solution of (12a) under the constraint (4). Note that its left-hand side contains the travel time derivative $t'(x)$, similarly to the refraction case. To find $t'(x)$, one fits the observed travel times with a cubic spline that is convex downward.

Because the data are fitted in each segment $[x_i, x_{i+1}]$ by a cubic parabola

$$T(x) = c_{i0} + c_{i1}(x - x_i) + c_{i2}(x - x_i)^2 + c_{i3}(x - x_i)^3,$$

where the constants c_{ij} ($j=0, 1, 2, 3$) are expressed in terms of the spline and its second derivatives at the grid points

$$\Delta : x_{\min} = x_0 < x_1 < \ldots < x_n = x_{\max},$$

the left-hand side of (12) can be written as

$$f(w) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \sin^{-1} \frac{c_{i1} + 2c_{i2}(x - x_i) + 3c_{i3}(x - x_i)^2}{w}\, dx -$$
$$- x_{\max}\sin^{-1}\frac{p_2}{w} + x_{\min}\sin^{-1}\frac{p_1}{w}.$$

To find a numerical solution of (12a), we divide the segment $[p_2, u_{\max}]$ into N subsegments $[u_{j-1}, u_j]$. Consider a jump function ΔH_j ($j=1, 2, \ldots, N-1$) on the segment $[p_2, u_{\max}]$. In this case we have N linear equations in N unknowns ΔH_j:

$$f(w_i) = \sum_{j=1}^{N} d_{ij}\,\Delta H_j, \quad i = 1, 2, \ldots, N, \tag{13}$$

where

$$d_{ij} = 2\ln\frac{(w_i^2 - p_2^2)^{1/2} + (u_j^2 - p_2^2)^{1/2}}{(w_i^2 - p_1^2)^{1/2} + (u_j^2 - p_1^2)^{1/2}} \tag{14}$$

is a positive definite matrix and $\Delta H_j \geq 0$.

Obviously, the solution of (13), which can be written in the form $A^T A \Delta H = A^T x$, minimizes the functional $S = \|A\Delta H - x\|^2$.

The problem is thereby reduced to quadratic programming in a finite-dimensional space and has a unique solution (provided the matrix $D = A^T A$ of (13) is nonsingular), because the functional S which is convex with respect to ΔH is bounded from below and is continuous on the convex set $\Re = \{\Delta H_j \mid \Delta H_j \geq 0\}$ [9].

Tackling (13), (14) by any of the known methods, for instance, by the method of conjugate gradients [8], we find the unknowns ΔH_j. The function $H(u)$ on $[p_2, u_{\max}]$ can be found from

$$H(u) = H_j + \Delta H_j, \quad u \in [u_j, u_{j+1}], \quad \Delta H_j \geq 0, \quad j = 1, 2, \ldots,$$
$$N-1, \quad H_1 = H(u^*) = 0.$$

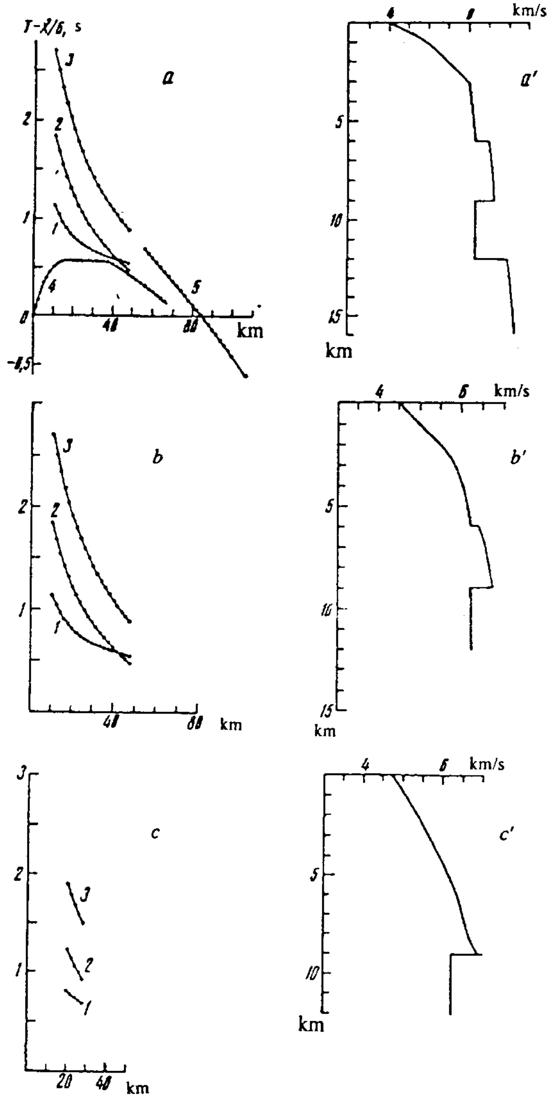

Figure 1 Travel time curves (*a*-*c*) and the respective velocity functions (*a*'-*c*'). 1-3 - fragments of reflection travel time curves; 4, 5 - fragments of reflection travel time curves. For explanation see the text.

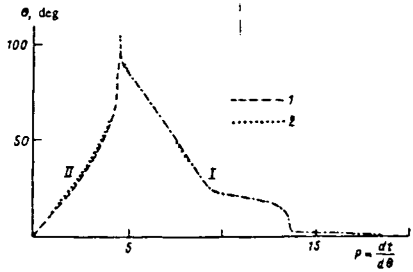

Figure 2 Variation of epicentral distance Q with ray parameter p for P (I) and PcP (II) times: 1 - after [8] and [9]; 2 - after [13], [23], and [27].

The end values p_2 and u_{max} can be improved by iteration. First, solve the problem for some $u^{*0}=p_2$ and $u_0^0=u_{max}$. Then take for u^* and u_0 those end values u_i and u_j for which ΔH_i and ΔH_j vanish. Next, solve the problem again for the end values u^{*1} and u_0^1, and so on until the end values ΔH_1 and ΔH_{N-1} vanish.

If, at some step of the iteration procedure, the values of u^* and u_0 happen to be close to one another, the matrix of (13) will be nearly singular. In that case the z_0z^* layer is to be treated as a layer of constant velocity whose parameters are found from the equations

$$x(p) = 2p\frac{\Delta z}{(u^2-p^2)^{1/2}}, \quad t(p) = 2\frac{u^2\Delta z}{(u^2-p^2)^{1/2}}$$

using the formulas

$$u = \frac{1}{p_2-p_1}\int_{p_1}^{p_2}(u(p)p)^{1/2}dp,$$

$$\Delta z = \frac{1}{2(p_2-p_1)}\int_{p_1}^{p_2}x(p)(u(p)/p-1)^{1/2}dp,$$

where $u(p)=t(p)/x(p)$.

Consider an example of calculation according to the above algorithm. Let the velocity structure consist of six layers, the velocity varying linearly in each. The velocity gradient is 1 s^{-1} in the first layer lying between depths 0 and 1 km. The second layer has a gradient of 0.5 s^{-1}. The third layer starts at depth 3 km and has a significantly lower gradient (0.07 s^{-1}). The velocity function has discontinuities at 6 and 9 km depths. The velocity gradient in the layer between these two depths is 0.07 s^{-1}. A waveguide of constant velocity is situated in the depth range 9-12 km with a third discontinuity at its lower boundary.

Figures 1, *a′* and 1, *a* present, respectively, a velocity function for this model and incomplete refraction and reflection travel time curves reduced with a velocity of 6 km/s. Rms 0.01 s random errors were added to the travel times, and the refracted and reflected data were interpreted jointly. The result was a curve that coincided almost exactly with the model curve.

Figure 1, *b* shows the results obtained by interpreting of reflection data alone. The curve is similar to the model. The depths to the discontinuities and the velocities above the reflectors were determined better than the velocities at the upper boundaries of the respective layers and the velocity in the region of a fast gradient variation.

Figures 1, *c* and ' *c'* show, respectively, short fragments of travel time curves and the velocity function obtained by inverting them. One can see that the positions of the boundaries are determined almost exactly, but the velocity function is smoother than the model curve.

Comparing the results of all three determinations, we can conclude that the best results are achieved with a joint interpretation of refraction and reflection data. Refraction data reveal regions of high velocity gradients but are less efficient at identifying segments of low gradient, because refracted waves do not penetrate deep into layers. Reflection data are efficient for locating interfaces and velocities above them.

Let us estimate the uncertainty of source depth location. Suppose the true depth is z^* and that we have deduced a depth of $\tilde{z}^*$ and found the velocity in the $z^*\tilde{z}^*$ layer to be constant. Then for $\delta t = \tilde{t} - t$ and $\delta x = \tilde{x} - x$ under the constraint $u(z)=u=$constant, $z\in[z^*, \tilde{z}^*]$ we have

$$\delta t = \frac{u^2}{(u^2-p^2)^{1/2}}\delta z, \quad \delta x = \frac{p}{(u^2-p^2)^{1/2}}\delta z.$$

Taking the quantity

$$\delta\tau(p) = ([\delta t(p)]^2 + [p\delta x(p)]^2)^{1/2},$$

as the deviation of the theoretical travel time curve $t=t(u)$ from the experimental curve $\tilde{t}=\tilde{t}(x)$, we have

$$\delta z(p) = \left[\frac{u^2-p^2}{u^4+p^4}\right]^{1/2} \delta\tau(p). \tag{15}$$

Formula (15) relates the error of z^* determination and $\delta\tau$, the error of epicentral distance and travel time along a ray of parameter p. If the solution is sought for the segment of the travel time curve bounded by the rays with parameters p_1 and p_2, then the error of z^* can be found as the mean of all values $p\in[p_1, p_2]$ by using

$$\begin{aligned}|\Delta z| &\leq \frac{1}{p_2-p_1}\int_{p_1}^{p_2} |\delta z(p)|\,dp = \frac{1}{p_2-p_1}\times \\ &\times\int_{p_1}^{p_2}\left[\frac{u^2-p^2}{u^4+p^4}\right]^{1/2}\delta\tau(p)dp \leq \frac{|\Delta\tau|}{u^2(p_2-p_1)}\int_{p_1}^{p_2}(u^2-p^2)^{1/2}dp = \\ &= \frac{1}{2u^2(p_2-p_1)}\left\{p(u^2-p^2)^{1/2} + u^2\arcsin\frac{p}{u}\right\}\Bigg|_{p_1}^{p_2}|\Delta\tau|,\end{aligned} \tag{16}$$

where $|\Delta\tau|\in[|\delta\tau|_{min}, |\delta\tau|_{max}]$.

In particular, for a complete travel time curve with $p_1=0$ and $p_2=u$, we have

$$|\Delta z| \leq \frac{\pi|\Delta\tau|}{4u}.$$

Table 1 *P*-wave velocity distribution in the mantle.

H, km	v_p	*H, km*	v_p	*H, km*	v_p
0.00	6.00	550	9.91	1800	12.55
14.90	6.00	600	10.17	1900	12.65
14.90	6.75	650	10.43	1000	12.78
22.49	6.84	700	10.74	2100	12.89
42.96	7.21	750	11.00	2200	13.01
42.96	8.05	800	11.07	2300	13.14
100.00	8.11	850	11.17	2400	13.24
150.00	8.20	900	11.26	2500	13.35
200.00	8.32	1000	11.45	2600	13.48
220.00	8.38	1100	11.60	2700	13.61
250.00	8.48	1200	11.77	2750	13.62
300.00	8.66	1300	11.90	2780	13.68
350.00	8.90	1400	12.05	2800	13.64
400.00	9.13	1500	19.19	2830	13.60
450.00	9.36	1600	21.31	2860	13.56
500.00	9.66	1700	12.43	2893	13.54

Numerical experiments showed that the results were in good agreement with the actual errors involved in the location of reflectors.

Determination of Velocity Structure in the Mantle and of the Radius of the Core. Velocity distribution in the mantle and the radius of the core were determined using unsmoothed *P* travel times obtained by Herrin *et al.* [23] and *PcP* travel times obtained by Taggart and Engdahl [27] and Kogan's [8], [9] smoothed *P* travel times.

The source data were examined by plotting epicentral distance θ as a function of ray parameter p. Figure 2 shows these functions. The Herrin *et al.* refraction times for epicentral distances of 20 to 105° were taken from [27], those for 0 to 20°, from [23] (Fig. 2 shows $\theta(p)$ for these travel times by dots). The rms smoothing error was 0.04 s. The times of *P* waves reflected from the core had been obtained by Taggart and Engdahl [27] for epicentral distances of 20 to 63°. The scatter of data points about the smoothing spline was 0.4 s.

The $\theta(p)$ curves based on Kogan's smoothed times (dashed lines in Fig. 2) showed that the *PcP* and *P* times were not mutually consistent in apparent velocity starting from the 95° distance; the *P* times were available for distances greater than 30°. For this reason calculations were made using *P* times for distances below 30° from [23], *P* times for 30 to 90° from [8], and *PcP* times for a 16-70° from [9]. The rms error of these times about the smoothed values was 0.7 s [9]. The *PcP* times were used below the 70° distance, because the "tail" of the travel time curve corresponding to reflections beyond the critical angle produced greater errors in the depth to the core than the initial part corresponding to subcritical reflections.

The depth to the core based on the times from [13], [27] was 2893 ± 2 km, which corresponds to a core radius of 3478 ±2 km; the value based on the times reported in [8], [9] was 2894 ± 1 km, which gave a radius of 3477 ± 1 km (error was determined here relative to the smoothed times). Velocities above the core boundary were 13.54 and 13.63 km/s, respectively. Comparison of these results with data from the literature showed the best agreement with the values of the radius of the core reported in [14] and [27]. The final velocity distribution in the mantle is given in Table 1.

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