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GEOPHYSICS

The Influence of Elastic Compressibility of the Mantle on Thermal-Gravitational Convection: The Convective Instability of the Gravitational Stress Pattern

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Abstract—The set of equations for the problem of thermal-gravitational convection accounts for compressibility of solid bodies, which changes for elementary volumes moving during convection process in fields of the initial temperature and the initial gravitational stress pattern for rheology of an elastic-viscous Maxwell body. It was shown that equations of momentum conservation in the vertical direction and heat transfer for steady convection differ from the equation for incompressible liquid by terms containing the rate of elastic volume change and the connected rate of heat change. It was established that an additional term in the momentum conservation equation defines a new class of the instable state of a solid body, which is able to form huge deformations at the expense of plasticity and creep at large segments of time—flow in the field of gravity force—instability of the gravitational stress pattern of elastic-plastic body. Analysis of different boundary conditions for which this instability can be realized in the form of convective cells showed that the convection rate is totally defined by reconstruction processes of vertical stresses on horizontal boundaries close to the initial gravitation pattern. Alignment process of these stresses can be provided not only by erosion and denudation processes occurring on the Earth's surface, but also by processes on the inner boundaries of the tectonosphere which provide isostasy in the mantle.

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INTRODUCTION

At present, convection in the mantle is calculated based on a model of the medium in the form of a viscous incompressible liquid with density ρ, which increases with depth. The problem is solved in the Euler representation for the excited stress-deformed state which accounts for the ability of the medium to change volume only within the framework of solving the thermo-elasticity problem [1, 2]. The initial gravitation stress pattern, deviation from which is defined within the thermo-elasticity problem, is presented through the initial distribution of all-around pressure P^0 [3]

$$P^0_{,z} = \rho(z)g \ \text{ at } \ \rho(z) = \rho_0(1-\theta^0), \qquad \theta^0 = -\frac{P^0}{K}, \quad \rho_0 = \rho(0) \tag{1}$$

with assumption of the absence of deviatory stresses. Here, θ^0 and K are the deformation of the elastic change in volume and the modulus of volume elasticity; correspondingly, the lower index after the comma means differentiation by the spatial coordinate and the coordinate system is connected with the surface of mantle and z axis directed to the center of the Earth.

Using Euler's coordinate system connected with a point of space allows considering a model of the medium in the form of an incompressible liquid with density corresponding to the initial unexcited state (1) in convection problems. With such a problem statement, momentum conservation equations do not account for elastic deformation of volume change caused by changing of the deep level of a substance during convection. In a classic monograph [4, p. 775] it is stated "nonuniform heating of a solid medium does not lead to the appearance of convection in it as usually takes place in liquids." Such a conclusion in this work was explained by the absence of comprehension about the possibility of formation of superlarge deformations, that is, fluid motion in solid bodies under high pressure and temperature at large times.

Real earth materials possess elasticity and the ability to change volume under change in the all-around pressure. Crystalline earth materials have another Poisson coefficient (ν), which varies in quite a wide range from 0.15 to 0.40. Data about the seismic wave rates in the core and mantle show that values of coefficient ν close to 0.25 exclude local anomaly regions, and the most contrasting of them are accumulated in

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the core. Change in the density of mantle material at the expense of increase of temperature (decreasing of density) does not exceed 5% (at temperature difference $DT = 2500$ K and the average value of thermal expansion coefficient $\alpha \approx 2 \times 10^{-5}$ K^{-1}). The remaining change in density is provided by pressure (average value $K \approx 5.5 \times 10^{11}$ Pa for the lower mantle.)

This work present defining equations for problem of thermal-gravitational convection of a solid body possessing elastic compressibility and the tendency toward viscous flow for large periods of time.

MOMENTUM CONSERVATION EQUATION OF THE THERMAL-GRAVITATIONAL CONVECTION PROBLEM FOR AN ELASTIC-VISCOUS BODY

We will use the simplest rheological model of the medium in the form of elastic (G is the constant elastic shear modulus) linear-viscous (η is the constant viscosity coefficient) Maxwell body that can take account of the formation of large irreversible deformations, as well as elastic compressibility. The final solution will be built for parameters of the excited state, which causes deviation of temperature as well as stress and deformation from the initial level (1) defined by the initial gradient of pressure (P^0) and temperature (T^0). It is assumed that the initial state by temperature corresponded to the adiabatic distribution law by depth $T^0 = T_s$. The Cartesian coordinate system of the problem defines the vertical location of axis r coinciding with the axis to the vertex.

Expressions connecting stresses (σ_{ij}) and deformation rates ($\dot{\varepsilon}_{ij}$) of the excited stress-deformed state with respect to the mentioned initial state for the elastic-viscous body subjected to large irreversible viscous deformations are denoted as

$$2\eta\dot{\varepsilon}_{ij} = \left(1 + \frac{\eta}{G}\frac{\partial}{\partial t}\right)\sigma_{ij} - \delta_{ij}\left(1 + \chi\frac{\eta}{G}\frac{\partial}{\partial t}\right)\sigma$$
$$+ \frac{2\eta\delta_{ij}}{3}(\alpha\dot{T}^0 + \dot{\theta}^{ug}), \quad i = x, y, r, \quad \chi = \frac{3\nu}{1+\nu}. \quad (2)$$

Here the dot above the function means partial derivative by time, δ_{ij} is the Kronecker symbol, and σ is the average stress. The last two terms in small round parentheses account for the influence of the temperature change rate ($\dot{T}^0$) and the volume change rate ($\dot{\theta}^{ug}$) because of the vertical displacement of the elementary volume in the initial temperature field T^0 and pressure P^0 on the rate of longitudinal deformations ($\dot{\varepsilon}_{ii}$); i.e., the influence of temperature ($T - T^0$) and pressure ($-\sigma$) excitation is neglected.

According to expressions (2) for the deformation rate, we have

$$\dot{\theta} = \frac{1}{K}\dot{\sigma} + \alpha\dot{T}^0 + \dot{\theta}^{ug} \ \text{ at } \ \dot{T}^0 \approx T^0_{,r}\dot{u}_r,$$
$$\dot{\theta}^{ug} \approx -\frac{P^0_{,r}\dot{u}_r}{K} = \frac{\rho g\dot{u}_r}{K}. \quad (3)$$

Momentum conservation equations will be denoted in the Euler coordinate system also for excited stress components taking account of the change in density at excited temperature T' ($T' = T - T^0$):

$$\sigma_{ij,j} + \delta_{ir}\alpha T'\rho g = \rho(\ddot{u}_i + \dot{u}_i\dot{u}_{i,j}), \quad i = x, y, r \quad (4)$$

after a series of standard transformations [4]. The momentum conservation equation in the vertical direction will be denoted by the desired rate $\dot{u}_r$ as

$$\eta\Delta(\Delta\dot{u}_r - \dot{\theta}_{,r}) + g\alpha\left(1 + \frac{\eta}{G}\frac{\partial}{\partial t}\right)\Delta_{xy}T'\rho = \rho(\ddot{u}_i + \dot{u}_i\dot{u}_{i,j}),$$
$$\Delta_{xy} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad g, \ \alpha = \text{const}. \quad (5)$$

Further we will consider the initial but already steady state of very slow flow. This allows neglecting products from excited displacement rate ($\dot{u}_i\dot{u}_{i,j}$) terms with a derivative by time from the displacement rate (acceleration) in (5) as well as considering the stress pattern as independent of time ($\dot{\sigma} = 0$). Expression (5) can be rewritten in the following way using (3):

$$\eta\Delta\left(\Delta\dot{u}_r - \frac{\rho g}{K}u_{r,r}\right) = -g\alpha\Delta_{xy}\rho T'. \quad (6)$$

In equation (6) the contribution to elastic compressibility of temperature was neglected.

Expression (6) is the first solution equation of the stated problem, which defines the momentum conservation law in the vertical direction for an elastic-viscous body possessing elastic compressibility and subjected to superlarge plastic deformations (flow) in the field of the initial stress pattern (1).

It is necessary to note that the momentum conservation equation itself in form (6) differs from a similar equation that can be obtained for the rheology of incompressible viscous liquid without accounting for elasticity, by an additional term in the left part. This term accounts for the elastic change in volume appearing in the medium at the expense of the vertical motion component in the field of the initial pressure gradient. In the development of the flow in the geological environment, one could consider that (6) corresponds to the rheology of a viscous elastic-compressible liquid.

In the right part of equation (6) there is a term with an unknown temperature. In order to complete the solution, it is necessary to prepare a heat transfer equation taking the peculiarities of this process in a solid elastic-compressible body into account.

Value of coefficient R_b for the main layers of the tectonosphere

Layer	H, km	ρg, g/cm^3	K, 10^5 kg/cm^3	R_b
Sedimentation pool	5–20	2.6	0.5–2	0.026–0.104
Core	40	2.7	7	0.015
Upper mantle	600	3.33–4.15	11–27	0.131
Lower mantle	2200	4.15–5.53	27–63	0.247

HEAT TRANSFER EQUATION FOR THE THERMAL-GRAVITATIONAL CONVECTION OF AN ELASTIC-VISCOUS BODY

For building the heat transfer equation, we will use the expression denoted in monograph [4] for solid body taking the possibility of appearance of superlarge irreversible deformations into account and as a result providing for the possibility of using Euler representations:

$$\rho T(\dot{S} + \dot{u}_i S_{,i}) = \kappa \Delta T + \sigma_{ij}\dot{u}_{i,j}$$
$$\text{at} \quad S = S^0 + K\alpha\theta. \qquad (7)$$

Here S and S^0 are the entropy of the final (at convection) and the initial (before convection) states, respectively, and κ is the thermal conductivity coefficient of mantle materials accepted as a constant in (7). An additional term in entropy compared to its initial value defines the change in the amount of heat of the solid body during expansion and compression.

Assuming the desired temperature T corresponding to the initial stage of convection, we can obtain the following expression instead of (7) for its small increment of adiabatic temperature ($T' \to 0$) given that it depends only on coordinate r:

$$\rho c_p \langle \dot{T}' + \dot{u}_i T'_{,i} \rangle$$
$$= \kappa \Delta T' - \dot{u}_r T_{s,r} - \frac{K\alpha T}{c_p}(\dot{\theta} + \theta_{,i}\dot{u}_i) + \sigma_{ij}\dot{u}_{i,j}. \qquad (8)$$

Here c_p is the thermal capacity coefficient.

In the Euler coordinate system, the influence of deformation of change in volume is accounted for at each point by the density measurement law that corresponds to the initial state, i.e., in (8) we can assume that $\theta = 0$. Using expression (3) and assuming that the flow process is steady (as for simplification of the momentum conservation equation) and the contribution to heat at the expense of internal viscous flow is small, instead of (8) we obtain

$$\rho c_p \langle \dot{T}' + \dot{u}_i T'_{,i} \rangle = \kappa \Delta T' - \left(T_{s,r} + \frac{\rho g \alpha}{c_p} T_s\right)\dot{u}_r. \qquad (9)$$

Equation (9) is the second solution equation of the thermal-gravitational convection problem which corresponds to heat transfer during flowing of an elastic-viscous body possessing elastic compressibility in the field of the initial gravitational stress pattern and temperature. In the right part of (9), there is an additional term in parentheses that was not present before when considering the mantle substance in the form of an incompressible liquid [2]. Its presence as for the additional term in momentum conservation equation (6) is connected with the influence of the initial gradient of pressure and the temperature of the compression process of solid materials. In the heat transfer equation (9), as for momentum conservation equation (6), the influence of temperature on compressibility was neglected. In addition, in the second term in parentheses, T was replaced by T_s.

Analysis of the terms in the right part of (9) connected with convection shows that for the lower mantle the sum of components in parentheses is positive. This is connected with the fact that there is too low a value of the adiabatic temperature gradient $T_{s,r} \approx -0.1$ K/km. Only in the upper mantle where a considerably higher value of this parameter holds $T_{s,r} \approx -0.6$ K/km is the sum of components in parentheses of the right part of (9) positive. The cooling down in the ascending convection branch of the lower mantle caused by elastic decompaction aligns the temperature of the rising materials with the surrounding environment. In the descending branch, elastic compaction leads to heating and decrease in speed of downwarping.

INSTABILITY OF THE GRAVITATIONAL STRESS PATTERN

It is necessary to note that the form of equation (6) allows defining the possibilities of loss of elastic-viscous layer stability, i.e., formation of the field of nonzero displacement rates in it in the case when the distribution of temperatures is lower than adiabatic. Assuming that $T^0 = 0$ and switching to dimensionless quantities in (6) by normalization of the coordinates to the power of layer H and rates to unit rate $\bar{\dot{u}}$, we can denote the solution of equation (6) in the form

$$\Delta(\Delta \dot{u}_r - R_b \dot{u}_{r,r}) = 0 \quad \text{at} \quad R_b = \frac{\rho g H}{K}. \qquad (10)$$

Here R_b is the coefficient whose value corresponds to the deformation of the elastic volume change under its own weight for a density-wise uniform solid body. Analysis shows that value of this coefficient for different tectonosphere layers (lower and upper mantle, the core on the whole, the main core layers and sedimentary cover) is almost always less than 0.3 (see table). At the same time, it decreases to values less than 0.01 with proceeding to upper layers of the tectonosphere.

It can be shown that at certain type of zero boundary conditions differential equation (10) will have a zero solution. As an example of solution of (10), we will consider the two-dimensional problem of flat deformation assuming that deformations and displacements along the y axis are zero. We will search for the solution (10) in the form

$$\dot{u}_r = C\exp(\beta r)\sin(\alpha x). \tag{11}$$

Here C is an unknown constant, $\alpha = 2\pi H/L$, where L is the horizontal size of the periodicity cell (two horizontal sizes of convection cell) and β is a dimensionless parameter.

Analysis of equation (10) accounting for inertial terms showed that at $\beta \geq \alpha$ the state of the rest of the layer becomes unstable, and at certain character of boundary conditions, there is no possibility for formation of a flow similar to convective flow. At the same time, the minimal value $\beta = \alpha$ corresponds to a state of flow totally controlled by processes taking place on surface layers. the considered type of convective flow will correspond to a loss of stability of the gravitational stress pattern conditioned by the presence of elastic compressibility in the geological environment, which is realized at a certain type of boundary conditions at the expense of the possibility of formation of large irreversible deformations.

The characteristic equation for differential equation (10) will have the following four roots:

$$\beta_1 = \beta_2 = \alpha, \quad \beta_{3,4} = \frac{1}{2}\left(R_b \pm \sqrt{R_b^2 + 4\alpha^2}\right). \tag{12}$$

Based on (11) and (12), the solution of equation (10) can be represented as

$$\dot{u}_r = [C_1\cosh(\alpha r) + C_2\sinh(\alpha r) + C_3\exp(\beta_3 r) + C_4\exp(\beta_4 r)]\sin(\alpha x). \tag{13}$$

Unknown integration constants C_i ($i = 1, 2, 3, 4$) must be defined from boundary conditions that are set for a layer infinite in terms of its upper and lower boundaries. When setting the zero boundary conditions for the layer (in terms of displacement rates or stresses), the possibility of a nonzero solution is defined by the equality of the discriminant of the algebraic equation set, obtained for realization of set boundary conditions in the form of a function of the displacement rate $\dot{u}_r$, to zero.

DEFINING THE INFLUENCE OF BOUNDARY CONDITIONS ON THE POSSIBILITY OF LOSS OF PATTERN STABILITY OF GRAVITATIONAL STRESS

When setting the mentioned boundary conditions corresponding to flat and slippery boundaries used in problems of thermal convection

$$\dot{u}_r = 0, \quad \sigma_{xr,x} = \Delta_\Gamma \dot{u}_r + \dot{\theta}_{,r} = 0 \quad \text{at} \quad \Delta_\Gamma = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial r^2} \quad \text{and} \quad r = 0, 1 \tag{14}$$

the discriminant of the algebraic equation set for unknown constants C_i is not zero at any $R_b \geq 0$.

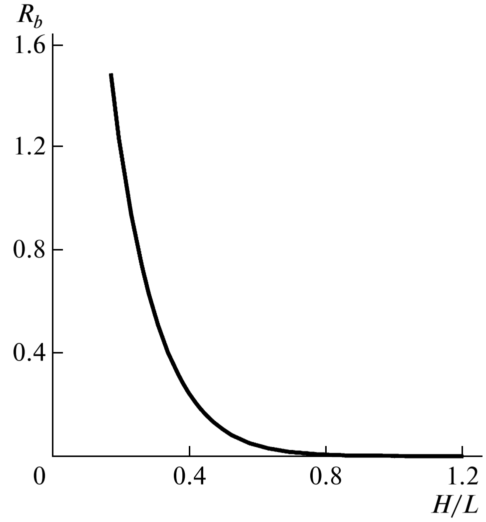

Fig. 1. Diagram of values of relationship H/L as a function of coefficient R_b at which loss of stability of the gravitational stress pattern of the elastic-viscous layer is possible.

Another variant of boundary conditions is presented by conditions that correspond to the nature of the main boundaries of the tectonosphere expressed in terms of excited desired functions:

$$\varepsilon_{rr} = \dot{\theta}^{ug}, \quad \sigma_{xr,x} = \Delta_\Gamma \dot{u}_r + \dot{\theta}_{,r} = 0 \quad \text{at} \quad r = 1, \tag{15}$$

$$\dot{u}_r = 0, \quad \sigma_{xr,x} = \Delta_\Gamma \dot{u}_r + \dot{\theta}_{,r} = 0 \quad \text{at} \quad r = 0. \tag{16}$$

This kind of conditions means solid and slippery lower boundaries and a free upper boundary at zero values of tangent (σ_{xr}) and normal (σ_{rr}) stresses, because for steady flow stages (2) and (4) imply

$$\sigma = -\frac{4}{3}\eta\dot{\theta}^{ug}. \tag{17}$$

The calculations performed showed that for conditions (15), (16) the discriminant of the set of linear algebraic equations for defining integration constants (13) has nonzero values at certain relationships between the coefficient R_b and the geometric parameters of layer H/L (Fig. 1).

It is necessary to note that conditions (15) are formulated for excited stresses and deformation rates that together with constancy of the initial stress pattern predefines the requirement of permanency of the upper boundary (the boundary for the layer must remain flat). The constancy condition of the initial shape of boundaries defines the necessity for external factors to perform additional work. First of all, such factors can include erosive and denudation processes taking place on the Earth's surface when considering sedimentary pools and the Earth's core on the whole as a layer subjected to convection. The same processes providing isostasy, i.e., intercore horizontal flowing of a substance, phase transformations on boundary core–upper mantle and upper and lower mantles, can

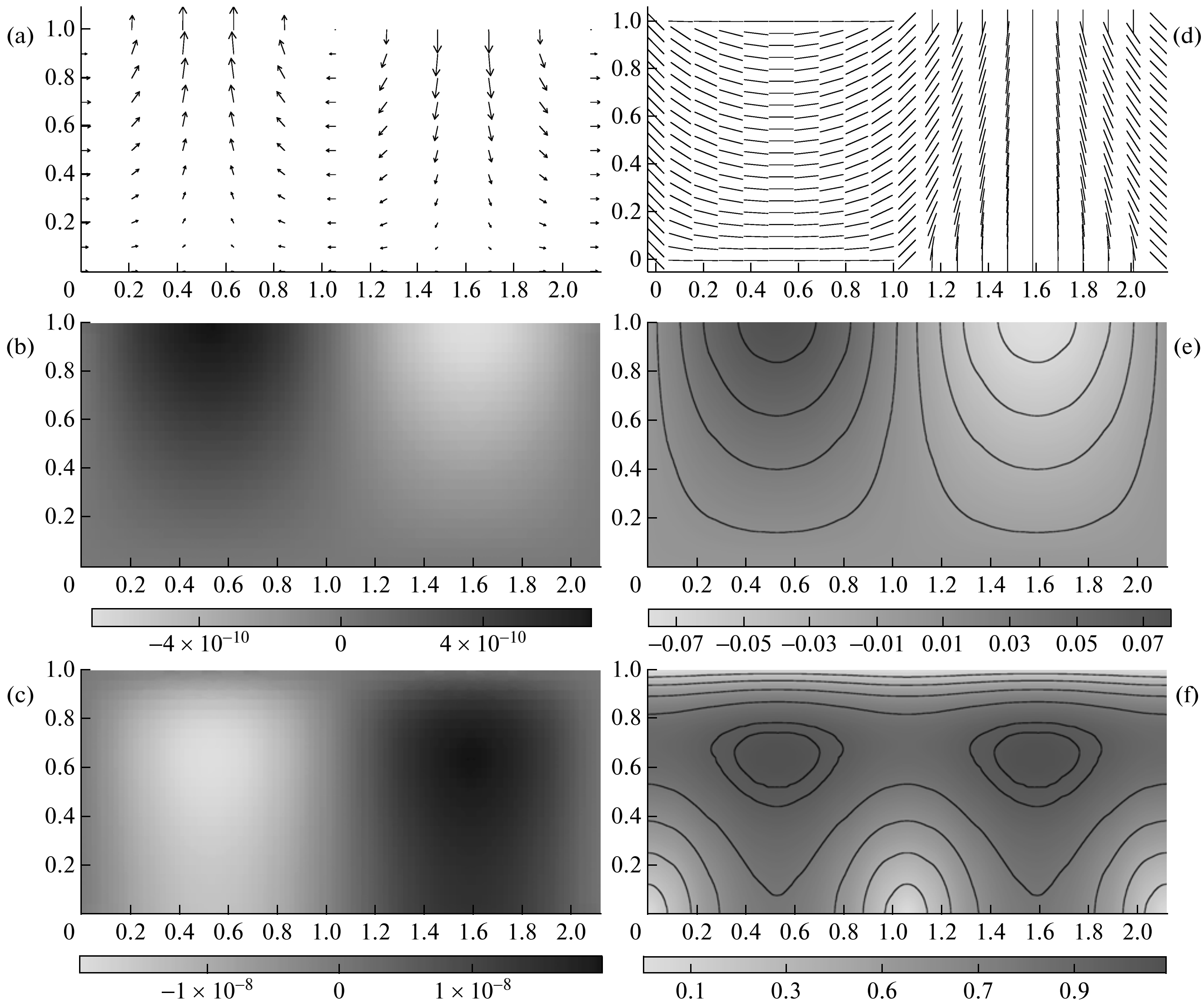

Fig. 2. Field of values of components of the stress-deformed state of the elastic-viscous layer subjected to convection caused by loss of stability of gravitational stress pattern: (a) vectors of displacement rates with lengths corresponding to displacement amplitudes (maximum 1 cm/year), (b) deformation rates of volume change ($year^{-1}$), (c) rates of longitudinal deformations in the horizontal direction ($year^{-1}$), (d) orientation of axes of maximal compression stress (MPa), (e) all-around pressure in the excited component, (f) maximal tangent stress (MPa). Geometric sizes are normalized to the power of layer *H*.

be considered as factors supporting closeness of the final stress pattern to the initial.

The analysis shows that the flow rate in the layer is totally controlled by the return rate of deformed boundaries to the flat state, i.e., by isostasy processes for the internal core and mantle layers, as well as erosion and denudation for surface layers. In fact these processes provide energy of convection process.

PECULIARITIES OF FLOW AND STRESS PATTERNS

Figure 2 shows calculation results of stresses and deformations for a layer with parameters of the upper mantle ($R_b = 0.1305$). It was assumed that the maximal values of the vertical rate on the upper surface are 1 cm/year. It can be seen in Fig. 2a that the orientation of the displacement rate considerably differs in the distribution of a similar field that can be obtained for thermal convection of an incompressible liquid for solid and slippery boundaries [5]. Convection caused by gravitational instability in the case of stress-free boundaries has an open cell lacking lateral transfer of the substance from the ascending branch to the descending one. This function is performed by those external factors that are responsible for isostatic compensation. The vertical component of the rate is maximal near the surface and horizontal in the middle part of the layer (depths 0.3–0.4 *H* from the surface).

Figures 2b and 2c show fields of deformation rates of volume changing and longitudinal deformation in the horizontal direction. The first ones have maximal values about $\pm 7.25 \times 10^{-10}$ and the latter, 2×10^{-8} year^{-1}. Such a relationship between these two types of deformation rates, which is typical for the given type of convection, is maintained also at other values of defining parameter R_b (for other layers of the tectonosphere).

In the orientation of stress axes of maximal compression (Fig. 2d), it is necessary to note that in the ascending branch of convection there is a horizontal compression mode and, in descending branch, a horizontal expansion mode. Such a character of axes is completely opposite to those that are shown by the results of thermal convection [6], data about stresses in subduction zones [7], and oceanic rift zones. However, such a distribution is in good agreement with the results of tectonophysical reconstruction of stresses for mountainy-infolded regions [8]. As a rule, maximal compression in the core of mountain uplifts is subhorizontal, and in sedimentation regions, these stresses are subvertical.

The level of deviatory stresses and all-around pressure (Figs. 2e, 2f) depends directly on the viscosity coefficient, which was accepted as 10^{21} Pa. The value of maximal tangent stresses reaches 1.1 MPa, and that of all-around pressure, 0.079 MPa, which is considerably less than the stress level in problems of thermal convection (up to 30 MPa [6]). The highest values of deviatory stresses are reached near the middle of the layer.

In this way the equations obtained for thermal-gravitational convection of an elastic body accounting for its elastic compressibility show the possibility of formation of convection in those regions of mantle where the gradient of over-adiabatic temperature exceeds the inverse gradient of temperature appearing during elastic volume change in the ascending and descending branches. Analysis of the obtained momentum conservation equations in the vertical direction showed that at constant temperature they define the possibility of new shape of the instable state which appears in the field of gravity force in a solid body possessing capability of flowing at large times. This form of instability, called the instability of the gravitational stress pattern of a solid body, is totally controlled by fulfillment of the closeness condition of vertical normal stresses to their values corresponding to the initial gravitational state on horizontal layers of the tectonosphere. Fulfillment of such conditions in the core is connected with erosion and denudation processes of the Earth's surface and in the mantle, with phase changing processes.

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