ISSN 1028-334X, Doklady Earth Sciences, 2010, Vol. 435, Part 2, pp. 1705–1709. © Pleiades Publishing, Ltd., 2010.
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GEOPHYSICS

Analytical Solution of the Problem for a Set of Shear Fractures with Coulomb Friction

Yu. L. Rebetskii and A. S. Lermontova

Presented by Academician V.V. Adushkin May 19, 2010

Received May 21, 2010

Abstract—A simplified solution of a two-dimensional problem on a set of interaction shear fractures with Coulomb friction along their faces is proposed. In the classical statement of this problem, the equality between tangential stresses and friction stresses must necessarily hold in a new equilibrium state at each point along the fracture plane. The solution of the problem is reduced to the solution of a set of singular integral equations with respect to unknown functions of a shear jump on fractures. In reality, our knowledge of rupture conditions is rather approximate. In addition, in the problems of seismology and tectonophysics it is sufficient to approximately estimate dynamic (taking into account the inertia forces) and static perturbations from a fracture in the form of the first terms of a true solution series. With this aim in view, point models of an earthquake source or continual representation of discontinuous deformations are used. Within the framework of the requirements of such approaches, it is assumed that the condition of equality between the sum of tangential stresses and friction stresses on the shear fractures in a new equilibrium state is met. In addition, the complex potential function is calculated for each fracture based on the function of the jump of its face shears obtained in the problem for a solitary fracture. Such statement of the problem of a set of neighboring and even intersecting fractures makes it possible to reduce its solution to a set of linear algebraic equations with respect to tangential stresses relieved at each fracture and the average along its length, they being the unknown values.

DOI: 10.1134/S1028334X10120317

Two statements of the elasticity theory problem on shear fractures is possible, namely specification of the function of the jump of fracture face shears (the theory of dislocation) and specification of stresses at infinity and the conditions of contact in the form of friction at the fracture faces. The first type of problem statement requires a priori specification of shears on fractures, which is a problem in the case of a set of interacting fractures. Within the framework of elasticity theory, the second type of problem statement for a set of neighboring fractures is reduced to the solution of a set of singular integral equations with respect to the unknown functions of a shear jump on fractures that requires large computational capabilities. In a recent paper, an approximate analytical solution of a two-dimensional problem on a set of interaction shear fractures with Coulomb friction along fracture faces is proposed. Specification of the type of function of the fracture face shear jump is a simplifying assumption. In this case, the shear amplitudes are the unknown values and their determination for a set of simultaneously or successively activated shear fractures is reduced to a set of linear algebraic equations. The statement of the problem in such a form is a combination of the two above-mentioned statements. Such an approach corresponds to proximity of our knowledge on the rupture conditions in rock massifs. In addition, in the problems of seismology and tectonophysics, it is often sufficient to approximately estimate dynamic (taking into account inertia forces) and static perturbations in the form of the first terms of the true solution series.

Rock massifs are a media containing many flaws in the form of earlier-formed and partially healed fractures. Under loading conditions, these fractures can again activate releasing a part of the internal energy accumulated in elastic deformations in the rock. Since a stressed state where compressing stresses are acting is formed in the rock almost everywhere at the depths of several hundred meters under the action of body forces (flanks of hills and open pit mines being an exception), activation of large fractures and ruptures with lengths of more than a few meters lies in the formation of fracture face shears.

In the theoretical studies of D. N. Osokina [1] for two-dimensional problems, it was shown that a perturbed stress field is generated in the vicinity of an activated shear fracture. It substantially distorts the initial homogeneous stressed state acting here prior to the fracture face shear. The dimensions of the perturbed

Schmidt Joint Institute of Physics of the Earth, Russian Academy of Sciences, Bol'shaya Gruzinskaya ul.10, Moscow, Russia
e-mail: reb@ifz.ru

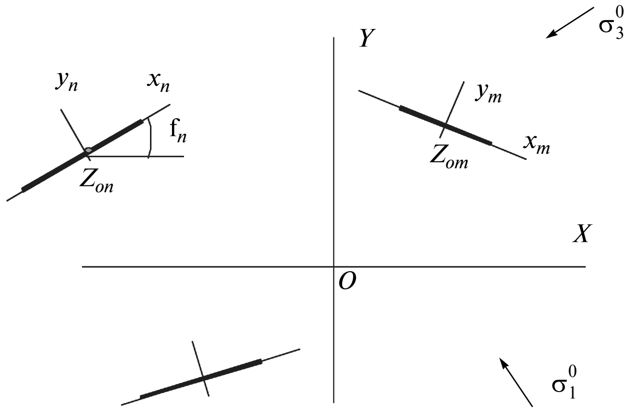

Fig. 1. Problem schemes.

state region make 3–5 fracture sizes, and this region has zones with a lowered (with respect to the initial state) level of deviatory stresses located along the middle part of the fracture and zones with an increased level of these stresses near the fracture ends. When investigating real crumbling rock massifs, we face both the problem of shear formation along the existing cracks or new ones formed under the conditions of an initial inhomogeneous stressed state and the problem of interactions of a set of simultaneously or successively activating fractures. The solution of such problems for cracks with a dry Coulomb friction in the classical statement as is the case in [2] faces a lot of difficulties when constructing analytical solutions and is actually reduced to a numerical solution of the problem with the help of finite-element or finite-difference methods.

PROBLEMS OF THE CLASSICAL SOLUTION IN THE THEORY OF FRACTURES

Consider a set $n = 1, 2, ..., M$ of neighboring shear fractures with the surface strength smaller than that of the massif beyond the fractures (Fig. 1). The two-dimensional problem of mechanics [4] with the initial inhomogeneous stressed state determined by the external conditions of loading and previous deformation stages is analyzed. A particular case of this initial state is the homogeneous state specified at an infinite distance by some quantities and the orientation of two principle stresses. We assume that this initial stressed state and the strength parameters along each fracture of the considered set are such that they predetermine simultaneous reaching of the limiting state after which fractures activate. Activation of each fracture lies in the appearance of shears of its faces in opposite directions without an increase in the initial fracture length and variations in the positions of its ends. The following limit equation

$$\left|\sigma^0_{xy}(x_n, y_n)\right| = \tau_s - k_s\sigma^0_{yy}(x_n, y_n) \text{ at } \quad |x_n| \le L_n, \quad y_n = 0 \text{ and } \sigma^0_{yy}(x_n, y_n) \le 0, \tag{1}$$

which specifies the interaction of tangential stresses with the surface strength limit along the entire fracture length, can be considered as one of the versions of such a limiting condition. Here τ_s and k_s are the limit of the surface adhesive strength and the coefficient of static surface friction, respectively, and $\sigma^0_{yy}(x_n, 0)$ and $\sigma^0_{xy}(x_n, 0)$ are the normal (negative compressing stresses) and, respectively, tangential stresses acting along the n-th fracture with the length of $2L_n$ till its activation and written in its local coordinate system (Fig. 1).

These shears yield a decrease in tangential stresses acting along the fracture faces. A new equilibrium state is determined meeting the following condition on each fracture

$$\left|\sigma^1_{xy}(x_n, y_n)\right| = -k_k\sigma^1_{yy}(x_n, y_n) \text{ at } |x_n| \le L_n, \quad y_n = 0, \tag{2}$$

which puts the stresses of kinematic surface friction $\sigma^1_{xy}(x_n, 0)$ in accordance with a new level of tangential stresses $k_k\sigma^1_{yy}(x_n, 0)$ $(k_k < k_s)$.

Relieved tangential stresses $\Delta\tau_n$ for the n-th fracture are determined as the difference of tangential stresses of the initial $\sigma^0_{xy}(x_n, 0)$ and the end $\sigma^1_{xy}(x_n, 0)$ stressed states:

$$\Delta\tau_n = \left|\sigma^0_{xy}(x_n, 0) - \sigma^1_{xy}(x_n, 0)\right| = \left|\sigma^0_{xy}(x_n, 0)\right| - k_k\sigma^1_{yy}(x_n, 0). \tag{3}$$

When solving the problem for such a set of simultaneously activated fractures, it is also possible to use complex Kolosov–Mushelishvili potentials [5] in the form

$$\Phi_n(z_n) = -\frac{i\mu}{(1+\kappa)\pi}\int_{-L_n}^{L_n}\frac{g'_n(r)}{r - z_n}dr. \tag{4}$$

Here, μ is the shear modulus, $\kappa = 3 - 4\nu$ and $\kappa = \frac{3-\nu}{1+\nu}$ are in the problem of the plane stressed and deformed state, respectively, where ν is the Poisson ratio. Its own complex potential $\Phi_n(z_n)$ will correspond to each fracture n, and $z_n = x_n + iy_n$ is the complex variable of the local coordinate system. This potential determines the contribution of the shear jump $g_n(x_n)$ along the faces of the given fracture to the total stressed state. The sought normal and tangential stresses are determined through these complex potentials [3]:

$$\sigma_{xx} + \sigma_{yy} = 2\left[\Phi(Z) + \overline{\Phi(Z)}\right] + \sigma^0_{xx} + \sigma^0_{yy},$$

$$\begin{aligned}&\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy}\\&= 2\left[\bar{Z}\Phi'(Z) + \Psi(Z)\right] + \sigma^0_{yy} - \sigma^0_{xx} + 2i\sigma^0_{xy}\end{aligned} \quad (5)$$

$$\text{at } \Psi(Z) = -2\Phi(Z) - Z\Phi'(Z) \text{ and } \Phi(Z) = \sum_{n=1}^{M}\Phi(Z).$$

Here, Z is a complex variable of the global coordinate system (Fig. 1) and the stresses with the superscript "0" correspond to the initial ones.

Application of Eqs. (4) and (5) together with condition (2) makes it possible to reduce the problem solution to a set of singular integral equations [3] with unknown shear jumps $g_n(x_n)$ for the analyzed system of fractures:

$$\begin{aligned}&-\frac{2\mu}{(1+\kappa)\pi}\int_{-L_n}^{L_n}\frac{g_n'(r)}{r - z_n}dr\\&= \operatorname{sgn}\langle\sigma^0_{xy}\rangle_n\left[\Delta\tau^0_n + \sum_{\substack{m = 1, 2, \dots, M\\ m\neq n}}\Delta\tau_{nm}\right]\\&\text{at } x_n \in L_n, \quad n = 1, 2, \dots, M.\end{aligned} \quad (6)$$

Here, $\Delta\tau^0_n$ in the right-hand side of Eq. (6) determines the relieved stresses along the crack n under the assumption that the neighboring fractures are at an infinite distance from it, and $\Delta\tau_{nm}$ characterizes the contribution into relieved stresses on the same fracture n owing to activation of the neighboring fraction m and contains an unknown shear jump $g_m(x_m)$. This parameter depends on the contribution of shears along the m-th fracture into tangential T_{nm} and normal N_{nm} stresses along the plane of the crack m

$$\begin{aligned}&\Delta\tau_{nm} = |T_{nm}| - k_s N_{nm} \text{ at } N_{nm} \le 0,\\&N_{nm} = 2\operatorname{Re}\left[\left(1 - e^{2i\phi_{nm}}\right)\Phi_m(Z_{nm})\right] +\\&+\operatorname{Re}\left[e^{2i\phi_{nm}}(\bar{Z}_{nm} - Z_{nm})\frac{d\Phi_m(Z_{nm})}{dZ_{nm}}\right]\end{aligned} \quad (7)$$

The coefficient $\operatorname{sign}\left\langle\sigma^0_{xy}\right\rangle_n$ in Eq. (6) takes into account the direction of the shear along the fracture. The variable Z_{nm} in Eqs. (7) is a coordinate of the point z_n belonging to the n-th fracture written in the local coordinate system associated with the m-th fracture. Moreover, ϕ_{nm} is the angle between positive directions of the axes x_n and x_m of the local coordinate systems x_nOy_n and x_mOy_m at the positive values counted counterclockwise from the x_m-axis (Fig. 1).

Since the complex potentials $\Phi_m(Z_{nm})$ in Eqs. (4) depend on the unknown fracture shear jumps, set of equations (6) contains singular integrals and the Fredholm integrals. This set of equations is difficult for constructing an analytical solution and requires application of numerical methods.

In the case of a single shear fracture with friction on its faces that activates in a homogeneous stress field, normal stresses before and after its activation remain constant, namely $\sigma^0_{yy}(x_n, 0) = \sigma^1_{yy}(x_n, 0) = \text{const}$ [6]. This very peculiarity of the condition on a fracture makes it possible to derive an analytical solution for a single fracture in a sufficiently simple form.

In the case of a set of simultaneously activating fractures, the normal stress on the fracture plane determining the level of tangential stresses in a new equilibrium state does not change owing to the self-shear of faces of each fracture. However, the existence of neighboring active fractures yields variations in these stresses. Thus, owing to crack interactions, normal stresses along their planes cannot be considered to be similar to those that acted at the initial stage ($\sigma^1_{yy}(x_n, 0) \neq \sigma^0_{yy}(x_n, 0)$ at $|x_n| \le L_n$). It should be noted that the condition for reaching a new equilibrium state at each fracture (condition (2)) depends not only on the shear amplitude of faces of this fracture but on shear on neighboring fractures as well. Only partial analytical solutions, e.g., two collinear fractures, are known for this problem statement [6] (solution of V.N. Friedman).

THE STATEMENT OF A SIMPLIFIED BOUNDARY-VALUE PROBLEM OF THE FRACTURE THEORY

The complexity of constructing an analytical solution for a set of simultaneously activating fractures is related to the appearance of integrals containing shear jumps along neighboring fractures in the right-hand side of set (6). It yields variability of normal stresses along the plane of each fracture and, as a result, tangential stresses at the stage of reaching a new equilibrium state. The effect of nonuniform distribution of stresses along the fracture plane on the solution result is related to selection of conditions on the fracture faces in the form of Eq. (2).

It is proposed to rewrite conditions (2) relating the process of reaching a new equilibrium state along the fracture faces with the integral conditions so that the sum of tangential stresses over the rupture area reached the sum of friction stresses on the same area. It means that conditions (2) are to be rewritten for average normal and tangential stresses acting along the fracture n

$$\left|\left\langle\sigma^1_{xy}(x_n, 0)\right\rangle_n\right| = -k_k\left\langle\sigma^1_{yy}(x_n, 0)\right\rangle_n \quad \text{at} \quad |x_n| \le L_n. \quad (8)$$

Triangular brackets $\langle\cdot\rangle_n$ mean that, instead of the bracketed quantity, we take its average value along the n-th fracture length. Here, in the general case we have $\left\langle\sigma^0_{yy}(x_n, 0)\right\rangle_n \neq \left\langle\sigma^1_{yy}(x_n, 0)\right\rangle_n$. Note that limit equation (1) can also be represented in the form of average values.

The condition for reaching a new equilibrium state along fractures in the form of Eq. (8) makes it possible

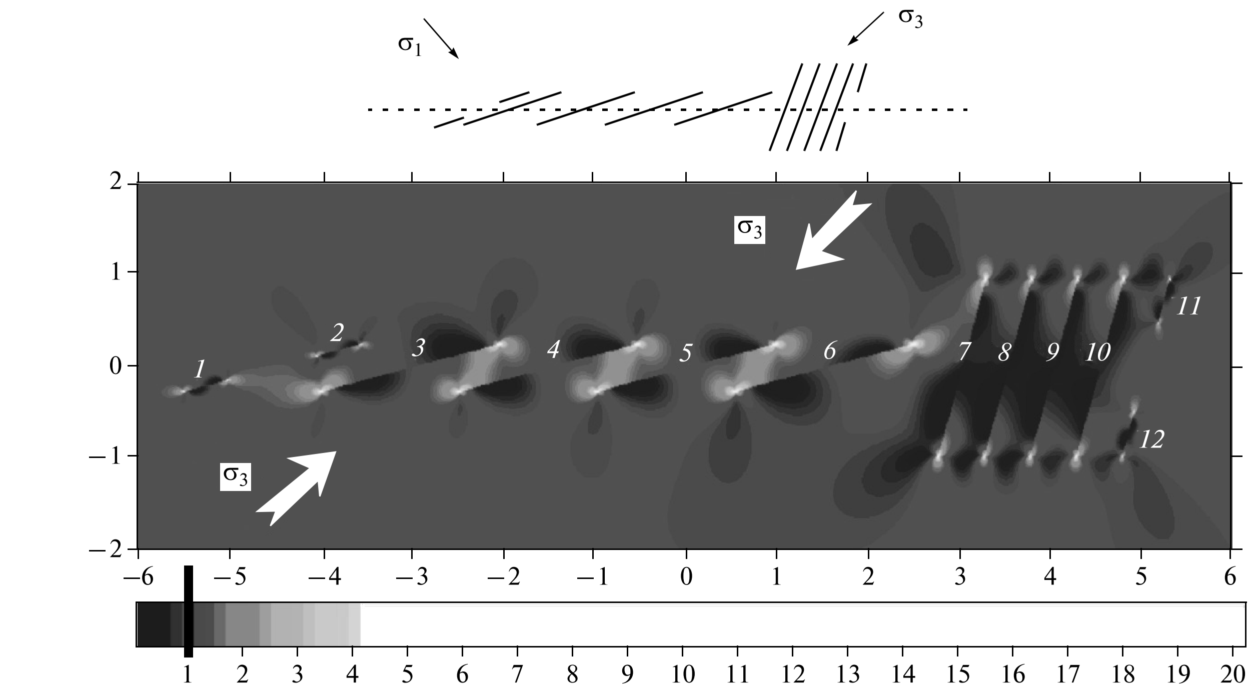

Fig. 2. An example of application of the presented approach to a system of fractures corresponding to the horizontal shifting zone (Riedel shear): (a) fracture positioning schematic (thin line is a shifting axis) and orientation of the axes of principle stresses of the initial homogeneous stress field; (b) the field of maximal tangential stresses in the zone of horizontal shifting (see comments in the text). A thin isoline corresponds to maximal tangential stresses, which are equal to 1 for the initial stress field (a vertical black dash on the value scale).

to specify the shear jump function and the complex potential in the form corresponding to the solution of the problem for a solitary fracture:

$$g(x_n) = \operatorname{sign}\left\langle\sigma_{xy}^0\right\rangle_n \frac{\Delta\bar{\tau}_n(1+\kappa)}{2\mu}\sqrt{L_n^2 - x_n^2} \quad \text{at } |x_n| \le L_n, \tag{9}$$

$$\Phi_n(z_n) = \operatorname{sign}\left\langle\sigma_{xy}^0\right\rangle_n \frac{i\Delta\bar{\tau}_n}{2}\left(1 - \frac{z_n}{\sqrt{z_n^2 - L_n^2}}\right).$$

Here, $\Delta\bar{\tau}_n$ are the unknown relieved tangential stresses along the n-th fracture.

In contrast to the above-mentioned classical solution, the unknown values in such problem statements are not shear jumps on the fracture but average relieved stresses. According to Eq. (8), let us integrate set (6) for each fracture n and, using Eqs. (7) and (9), come to a set of linear algebraic equations for the unknown $\Delta\bar{\tau}_n$:

$$\Delta\bar{\tau}_n = \operatorname{sgn}\langle\sigma_{xy}^0\rangle_n\left[\Delta\tau_n^0 + \frac{1}{2L_n}\sum_{m=1,2,\ldots,M}^{m\neq n}\Delta\bar{\tau}_m \times\left\{\operatorname{Re}\left[e^{2i\phi_m}(2L_n - A_{nm}) - \frac{B_{nm}+A_{nm}}{2}\right] - k_k\operatorname{Im}\left[(1-e^{2i\phi_{nm}})(2L_n - A_{nm}) + \frac{B_{nm}+A_{nm}}{2}\right]\right\}\right], \tag{10}$$

at

$$A_{nm} = e^{-i\phi_{nm}}[F_m(L_n) - F_m(-L_n)],$$

$$B_{nm} = e^{i\phi_{nm}}\left(\frac{L_m^2 - \bar{Z}_{nm}(L_n)Z_{nm}(L_n)}{F_m(L_n)} - \frac{L_m^2 - \bar{Z}_{nm}(-L_n)Z_m(-L_n)}{F_m(-L_n)}\right),$$

$$F_m(\pm L_n) = \sqrt{Z_{nm}^2(\pm L_n) - L_m^2},$$

$$Z_{nm}(\pm L_n) = (-Z_{mn}^0 \pm L_n e^{i\phi_n})e^{-i\phi_m}.$$

Having obtained relieved stresses for each fracture from set (10) and using the sum function of the potential $\Phi(Z)$ (see Eq. (5)), it is possible to calculate the parameters of a perturbed stressed state formed owing to simultaneous activation of a set of shear fractures.

In case fractures are activated successively rather than simultaneously (after checking for modified in the form of condition (1) on average)), Eq. (10) makes it possible to calculate relieved tangential stresses on each of these fractures.

Simplifying statements for the Coulomb correlations written along the fractures for average stresses are used at present in seismology for estimating dynamic parameters of earthquakes within the framework of a source point model [8]. In addition, we have approximate ideas on the conditions at the faces of fractures that are tens and thousands of kilometers long. Sometimes it is sufficient in the first approximation to evaluate the effect of ruptures on the stressed state of the

Earth's subsurface, and, thus, we replace rupture shears with continual deformations. As far as the study of the mutual effect of shear ruptures in natural objects (foreshock and aftershock sequences) and the study of tectonics of strongly crumbled geological objects are concerned, the proposed solution probably meets the requirements to the accuracy of the source data on ruptures and the data on the initial field of tectonic stresses.

As an example of implementation of the proposed approach, Fig. 2 shows the results of calculations of maximal tangential stresses for a train of shear fractures forming in the regions of horizontal shifting (*R* and *R*', Riedel shear). The initial stressed state was homogeneous with a 45° orientation with respect to shifting and was characterized by the following values: $\sigma_3^0 = -3$, $\sigma_1^0 = -1$, the kinematic friction coefficient was taken as zero. The fractures have two different sizes, namely $L_1 = 2$ and $L_1 = 0.5$, and the angles with a shifting axis were 15° and 75°, respectively. It is apparent from Fig. 2 that the zones of location of different types of fractures (R is fractures of the left shear, *R* is fractures of the right shear) are the regions of lowered maximal tangential stresses with respect to the initial stress fields (*R*' shear). However, at a certain ratio of their positions and distances between them, there appear vast sections of increased deviatory stresses (*R*-shears) in these zones. The relieved stresses for a fracture of a smaller size and the contribution to the perturbed state depend on their positioning with respect to relatively larger fractures. The average tangential stresses τ relieved on each fracture are listed in the table:

No.	1	2	3	4	5	6
τ	–1.01	–0.79	–0.90	–0.88	–0.92	–1.04
№	7	8	9	10	11	12
τ	0.88	0.73	0.72	0.85	0.66	0.96

Solution (1) presented in the present paper corresponds to a symmetric shear jump on each fracture (see Eq. (9)), and we consider it as the first term in the series the sought stressed state perturbed by activating fractures can be expanded to. However, this approach makes it possible to take into account the second term of the series responsible for skew symmetry by introducing a term with a linear dependence on the coordinate to the equation for a shear jump. Our analysis shows that these two terms are sufficient for investigating the stressed state of crumbled massifs for the problems of seismology and hydrophysics.

ACKNOWLEDGMENTS

This study was supported by the Russian Foundation for Basic Research, project nos. 09-05-012213a and 09-05-01022a.

REFERENCES

1. D. N. Osokina, *Stress and Deformation Fields in the Earth's Crust*(Nauka, Moscow, 1987) [in Russian].
2. B. V. Kostrov, *Mechanics of a Tectonic Earthquake Source* (Nauka, Moscow, 1975) [in Russian].
3. Yu. L. Rebetsky, D. N. Osokina, V. V. Ektov, *Tectonophysics Today* (OIFZ, Moscow, 2002) [in Russian].
4. V. V. Panasuk, *Limit Equilibrium of Brittle Bodies with Cracks* (Naukova Dumka, Kiev, 1968) [in Russian].
5. N. I. Mushelishvili, *Some Basic Problems of Mathematical Theory of Elasticity. Basic Equations. Plane Theory of Elasticity. Twisting and Bending* (Nauka, Moscow, 1966) [in Russian].
6. D. N. Osokina, V. N. Freedman, *Stress and Deformation Fields in the Earth's Crust* (Nauka, Moscow, 1987) [in Russian].
7. G. A. Sobolev, A. V. Ponomarev, *Earthquake Physics and Forerunners* (Nauka, Moscow, 2003) [in Russian].
8. J. Rice, *Mechanics of an Earthquake Source. Mechanics Series. New in Foreign Science* (Mir, Moscow, 1981) [in Russian].

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