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Stress State of Layer with Longitudinal Shear

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A study was made of the process of deformation of a viscous layer caused by horizontal displacement of one block of its basement relative to another along a fault separating them. The stress state in the layer and the kinematics of the surface are determined and the trajectories of the main normal and shear stresses are retrieved. The theoretical solution which is found, corresponding to the initial stage in deformation of the sedimentary layer of the crust with a corresponding movement of blocks of the crystalline basement, makes it possible to supplement and explain known concepts concerning the nature of the process transpiring during longitudinal shear obtained in experiments in models of equivalent materials.

The problem of equilibrium of a ponderable layer of constant thickness H of a linearly viscous incompressible material lying on a rigid basement broken by a plane fault is examined. It is assumed that the fault is an infinitely narrow slit and there is total adhesion between the bottom of the layer and basement blocks. The upper boundary of the layer is considered load-free. The rates of movements and the stress field arising in the layer as a result of the translational movement of basement blocks relative to one another at the constant rate W_0, whose direction is parallel to the trace of the fault on the plane of contact between the layer and the basement, are investigated. Under such conditions the layer is in a state of horizontal longitudinal shear. Such a problem, but with a vertical relative rate of movement of the blocks of the rigid basement, was investigated in [1].

In the solution we will assume that the rate of relative movement of the blocks W_0 and the time of examination are so small that it is possible to neglect the change in the form of the layer, not taking into account the inertial force and the difference between the Euler and Lagrangian coordinates.

If the layer is referenced to an arbitrarily fixed right-hand coordinate system XYZ (see Fig. 1) and if the Z axis is directed along the line of intersection of the planes of the bottom of the layer and fault, from the condition that the rate of relative movement of the blocks has a translational character it follows that there is a nondependence of the components of the tensor of stresses and the velocity vector of the rate of movements at any point in the layer on the Z coordinate. It can be shown that in such a formulation reference is to the problem of antiplane deformation for a viscous layer with the corresponding boundary conditions.

We will seek a solution in dimensionless form, assigning the stress dimensionality values to $4\eta|\dot{W}_0|/H$ velocity to $|\dot{W}_0|$, length to H. Here η is the viscosity of the layer material. The principal solution equation for the problem of antiplane deformation is a harmonic differential equation for the dimensionless velocity of movements

$$\nabla^2 \dot{w}(x, y) = 0, \tag{1}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a Laplace operator and the dot over w denotes time differentiation. Equation (1) must be solved under the boundary conditions

$$y=1:\ p_{yz}=0;\ y=0:\ \dot{w}=\tfrac{1}{2}\operatorname{sign} x, \tag{2}$$

where p_{yz} is dimensionless shear stress. In writing the latter boundary condition it was taken into account that relative to the selected fixed coordinate system the basement blocks and therefore also the bottom of the layer move in different directions with velocities equal to $|\dot{W}_0|/2$; the right block (see Fig. 1) moves in the direction of positive z.

We will represent the sought-for solution—the dimensionless velocity of movements $\dot{w}(x, y)$—as

$$\dot{w}(x, y) = \int_{-\infty}^{\infty} F(\alpha, y) \sin \alpha x \, d\alpha. \tag{3}$$

For dimensionless shear stresses p_{xz} and p_{yz}, representing the partial derivatives of $\dot{w}$ along x and y respectively, from (3) we obtain

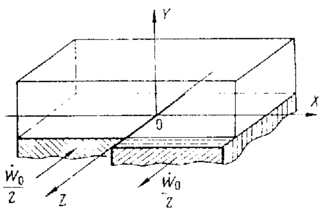

Fig. 1. Diagram of longitudinal shear problem.

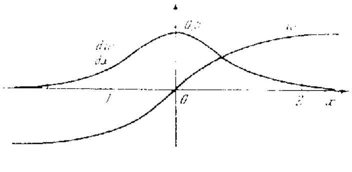

Fig. 2. Curves of dimensionless rate $\dot{w}$ and its gradient $\partial\dot{w}/\partial x$ at layer surface.

$$p_{xz}=\frac{1}{4}\int_{-\infty}^{\infty}\alpha F(\alpha, y)\cos\alpha x\, d\alpha,$$
$$p_{yz}=\frac{1}{4}\int_{-\infty}^{\infty}F'(\alpha, y)\sin\alpha x\, d\alpha. \tag{4}$$

Here $F(\alpha, y)$ is a function subject to determination; the prime denotes differentiation for y. Substituting (3) into (1), for $F(\alpha, y)$ we obtain the ordinary differential equation

$$F''(\alpha, y)-\alpha^2F(\alpha, y)=0,$$

whose general integral has the form

$$F(\alpha, y)=C_1\,\mathrm{sh}\,\alpha y+C_2\,\mathrm{ch}\,\alpha y.$$

In order to ascertain C_j (j = 1, 2), unknown functions of α, it is necessary to have recourse to the boundary conditions (2). The first of these, for the stresses p_{yz}, can be applied directly to (4); in order to use the second it is first necessary to write the function $\dot{w}(x, y)$ through an integral Fourier representation in the form

$$\dot{w}(x, 0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\alpha\int_{-\infty}^{\infty}\dot{w}(\lambda, 0)\cos\alpha(\lambda-x)\,d\lambda.$$

As a result, we find that

$$C_1=-\frac{\mathrm{sh}\,\alpha}{2\pi\alpha\,\mathrm{ch}\,\alpha},\quad C_2=\frac{1}{2\pi\alpha}$$

and therefore

$$F(\alpha, y)=\mathrm{ch}\,\alpha(1-y)/(2\pi\alpha\,\mathrm{ch}\,\alpha).$$

Then for $\dot{w}(x, y)$, on the basis of (3), we will have the following expression

$$\dot{w}(x, y)=\frac{1}{\pi}\,\mathrm{arctg}(\mathrm{sh}(\pi x/2)/\sin(\pi y/2)).$$

For the dimensionless shear stresses p_{xz} and p_{yz}, from (4) we obtain:

$$p_{xz}=\frac{\sin(\pi y/2)\,\mathrm{ch}(\pi x/2)}{4(\mathrm{ch}\,\pi x-\cos\pi y)},\quad p_{yz}=-\frac{\cos(\pi y/2)\,\mathrm{sh}(\pi x/2)}{4(\mathrm{ch}\,\pi x-\cos\pi y)}. \tag{5}$$

Turning to the kinematics of the free surface of the layer, we will also cite expressions for $\dot{w}$ and $\partial\dot{w}/\partial x$ when y = 1:

$$\dot{w}(x, 1)=\frac{1}{\pi}\,\mathrm{arctg}\left(\mathrm{sh}\,\frac{\pi}{2}x\right),\quad \frac{\partial\dot{w}(x, 1)}{\partial x}=\frac{1}{2}\,\mathrm{sech}\,\frac{\pi}{2}x.$$

Figure 2 shows $\dot{w}(x, 1)$ and $\frac{\partial}{\partial x}\dot{w}(x, 1)$ curves. It can be noted that the region of substantial change of these functions, outside which they are virtually equal to their asymptotic values, occupies a zone of about six layer thicknesses; the $\frac{\partial}{\partial x}\dot{w}(x, 1)$ maximum is situated on the y axis. As follows from (5), at the bottom of the layer

$$p_{yz}(x, 0)=-\frac{1}{8\,\mathrm{sh}(\pi x/2)},\quad p_{xz}(x, 0)=\delta(x),$$

where $\delta(x)$ is a delta function. Thus, the stresses p_{xz} and p_{yz} have a singularity at the origin of coordinates. These singularities in the stresses, being a result of a discontinuity of the first kind in the velocities of movements stipulated at the bottom of the layer, are nonintegrable, which makes it impossible to determine the magnitude of the force imparted to the bottom which governs the shear mechanism. We will therefore regard the constructed solution as asymptotic.

For the dimensionless main normal and shear stresses, taking into account hydrostatic pressure

$$p_x=p_y=p_z=-K_1(1-y), \tag{6}$$

where $K_1=\rho gH/4\eta|W_0|$, ρg is the specific weight of layer material, and that $p_{xy}(x, y)$ = 0, we obtain

$$p_1=p_2+p_{13},\ y_3=p_2-p_{13},\ p_2=-K_1(1-y),$$
$$p_{13}=\frac{1}{4\sqrt{2}(\mathrm{ch}\,\pi x-\cos\pi y)},\quad p_{12}=p_{23}=\frac{1}{2}p_{13}. \tag{7}$$

Figure 3a, with allowance for symmetry relative to the y axis, shows isolines of the main shear stresses p_{13} in a vertical section of the layer. With x = 0 and y = 0 the p_{13} stresses have the

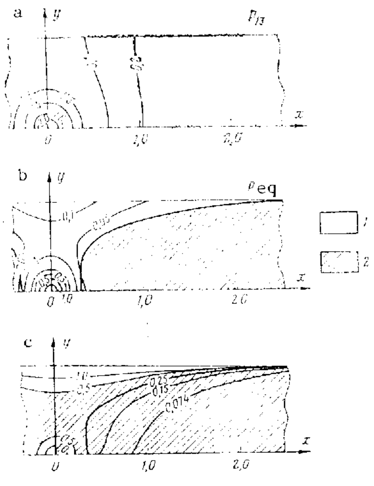

Fig. 3

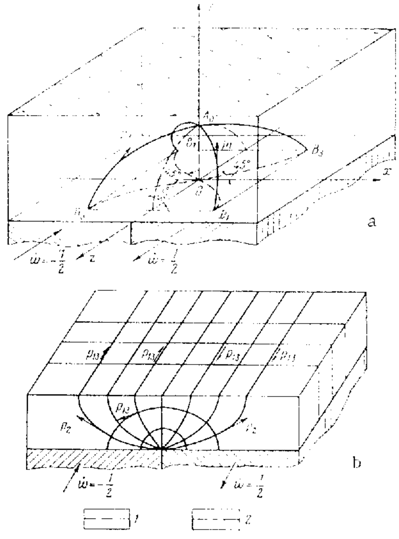

Fig. 4

Fig. 3. Isolines of dimensionless stresses: a) main shear stresses p_{13}, b) Mohr-equivalent stresses p_{eq}, c) zero isolines of main stress p_1 for different K_1 values; *1*) $p_1 p_3 < 0$; *2*) $p_3 < p_1 < 0$.

Fig. 4. Trajectories and surfaces of operation of vectors of main stresses: a) $\mathbf{p}_1$ and $\mathbf{p}_3$ in volume of layer, b) $\mathbf{p}_2$ and $\mathbf{p}_{13}$ in transverse section of layer; *1*, *2*) projections of trajectories of $\mathbf{p}_1$ and $\mathbf{p}_3$ vectors onto bottom of layer and vertical section coinciding with *yoz* plane.

earlier noted singularity. The main stresses were computed for values of the dimensionless coefficient, taking hydrostatic pressure into account, $K_1 = 0.25$. The parameters determining the K_1 value were taken in accordance with the parameters of the model and conditions for modeling the longitudinal shear used by A. V. Mikhaylova at the Institute of Earth Physics, USSR Academy of Sciences: $\rho g = 1.7$ g/cm^3, $H = 10$ cm, $\dot{W}_0 = 1$ mm/min, $\eta = 10^8$ P.

In comparing the stresses p_1 and p_3 we note that with a given K_1 in a quite broad zone directly over the fault there is a difference in the signs on these stresses; here, however, with approach to the fault there is also a sharp increase in shear stresses p_{13}. Adhering to the Mohr strength hypothesis [2] and the Coulomb-St. Venant hypothesis, impairments in continuity must be expected primarily precisely in this zone, that is, where p_3 and the product $p_1 p_3$ are less than zero. Shear fractures will develop here. The Mohr strength condition can be expressed through the values of some dimensionless "equivalent" stresses using the formula

$$p_{eq} = p_1 - kp_3 \leq \sigma_{np} H^2/K \text{ when } p_1 p_3 < 0.$$

Figure 3b shows isolines of equivalent stresses for values $k = 0.1$; in a region where p_1 and p_3 are of the same sign, p_{eq} was computed using the formula $p_{eq} = -kp_3$. The discriminated region corresponds to a value $p_1 p_3 > 0$. Figure 3c shows the interfaces of zones with $p_1 p_3 > 0$ and $p_1 p_3 < 0$ for different values of the coefficient $k_1 = 0.074$, 0.15, 0.25, 0.5, 1.0 (the region where $p_3 < p_1 < 0$, with $K_1 > 0.5$ is shaded).

During recent years various methods for retrieving the trajectories of the main normal stresses in the sedimentary layer of the crust [2-5], based on an analysis of orientation of shear surfaces and directions of movements along them, obtained using geological and seismic data, have come into wide use in geotectonics. This also makes it possible to use the pattern of trajectories of the vectors of the main normal stresses for clarifying the mechanism of formation of different geological structures, together with the velocities of movements measured at the surface [1].

For determining the direction cosines of these vectors we have the following expressions:

$$(p_i-p_x)l_i-p_{xz}n_i=0,\quad l_i^2+m_i^2+n_i^2=0,$$
$$(p_i-p_y)m_i-p_{yz}n_i=0,$$
$$p_{xz}l_i-p_{yz}m_i+(p_i-p_z)n_i=0,\quad i=1,2,3.$$

Here l_i, m_i, n_i are the direction cosines of the vector of the main normal stress p_i with the coordinate axes x, y, z, respectively, substituting here the p_i values taken from (7) and p_x, p_y, p_z, p_{xz}, p_{yz} from (6), (5), we obtain:

$$l_i=(-1)^{\frac{i-1}{2}}\frac{\sin(\pi y/2)\operatorname{ch}(\pi x/2)}{\sqrt{\operatorname{ch}\pi x-\cos\pi y}},$$
$$m_i=(-1)^{\frac{i+1}{2}}\frac{\cos(\pi y/2)\operatorname{sh}(\pi x/2)}{\sqrt{\operatorname{ch}\pi x-\cos\pi y}},$$
$$n_i=\frac{\sqrt{2}}{2},\quad i=1,3, \tag{8}$$
$$l_2=\sqrt{2}\frac{\cos(\pi y/2)\operatorname{sh}(\pi x/2)}{\sqrt{\operatorname{ch}\pi x-\cos\pi y}},$$
$$m_2=\sqrt{2}\frac{\sin(\pi y/2)\operatorname{ch}(\pi x/2)}{\sqrt{\operatorname{ch}\pi x-\cos\pi y}},\quad n_2=0.$$

In turn we can write

$$l_i=\frac{dx_i}{dt_i},\quad m_i=\frac{dy_i}{dt_i},\quad n_i=\frac{dz_i}{dt_i},\quad i=1,2,3, \tag{9}$$

where t_i is a curvilinear coordinate, reckoned along the trajectory arc of the main normal stress p_i, and x_i, y_i, z_i are the Cartesian coordinates of the points on this curve. It follows from (9) that

$$dx_i/dy_i=l_i/m_i,\ dy_i/dz_i=m_i/n_i,\ dx_i/dz_i=l_i/n_i. \tag{10}$$

In order to write equations defining the trajectories of the main normal stresses $\mathbf{p}_1$ and $\mathbf{p}_2$, we will integrate the first two expressions from (9). We obtain equations for two cylindrical surfaces with generatrices parallel to the z and x axes which with their intersection also give the sought-for curves. Since with integration two integration constants will enter into the derived expressions, for their determination we will assume that the trajectory must pass through the stipulated point A_0 on the plane xoy with the coordinates $x_0 = 0$, y_0 (see Fig. 4a). Thus, the equations for cylindrical surfaces assume the form

$$y_i=\frac{2}{\pi}\arccos\left(\cos\frac{\pi}{2}y_0\operatorname{ch}\frac{\pi}{2}x_i\right),$$

$$z_i=(-1)^{\frac{i-1}{2}}\frac{2a}{\pi}|R(\beta,a)-R_n(a)|,\quad i=1,3, \tag{11}$$

where

$$a=\sin\frac{\pi}{2}y_0,\quad \beta=\arcsin\left(\frac{\sin(\pi y_0/2)}{\sin(\pi y_i/2)}\right),$$

and $R(\beta, a)$ is an elliptical and $R_n(a)$ is a full elliptical integral of the first kind [6]

$$R(\beta,a)=\int_0^\beta\frac{d\lambda}{\sqrt{1-a^2\sin^2\lambda}},\quad R_n(a)=R\left(\frac{\pi}{2},a\right).$$

Expressions (11) are correct for $y_0\neq 1$. with $y_0 = 1$ in place of them we will have

$$z_i=(-1)^{i-1}x_i, \tag{12}$$

that is, if point A_0 lies on the surface, the trajectories $\mathbf{p}_1$ and $\mathbf{p}_3$ degenerate into straight lines lying in the plane xoz and forming a 45° angle with the z-axis. By varying the y_0 values in (11) from zero to unity (when $y_0 = 1$ it is necessary to use (12)), it is possible to construct a family of $\mathbf{p}_1$ and $\mathbf{p}_3$ trajectories passing through points which lie along the y axis. Since the planes parallel to the plane xoy are equally justified and the orientation of the vectors of the main stresses is not dependent on z (8), by simple transfer along the z axis it is also possible to obtain a family of trajectories passing through all the points on the plane yoz and, accordingly, also through all the points of the layer. The trajectories $\mathbf{p}_1$ and $\mathbf{p}_3$, passing through the stipulated point A_0 (0, 0.5, 0), were constructed in Fig. 4a. The figure indicates that the $\mathbf{p}_1$ trajectory has two points of intersection with the bottom of the layer. At these points the tangent to the curve, representing the $\mathbf{p}_1$ vector, is parallel to the plane yoz and forms an angle 45° with the Z axis. At point A_0 the $\mathbf{p}_1$ vector is parallel to the plane xoz and also forms a 45° angle with the z axis. The $\mathbf{p}_3$ trajectory, passing through the A_0 point, is a mirror reflection of the $\mathbf{p}_1$ trajectory relative to the yoz plane.

As follows from (8), the $\mathbf{p}_2$ trajectories are plane curves ($n_2 = 0$), lying in planes parallel to the yox plane. Integrating the first expression (10), with an accuracy to the integration constant we obtain an equation for the trajectory $\mathbf{p}_2$. The unknown constant is determined from the condition of passage of the curve through a stipulated point on the xoy plane with the coordinates x_0, $y_0 = 1$:

$$y_2 = \frac{2}{\pi} \arcsin(\text{sh}(\pi x_2/2)/\text{sh}(\pi x_0/2)), \quad (13)$$

and sign x_2 = sign x_0. Using the nondependence of orientation of the $\mathbf{p}_2$ vector on z and varying x_0 from $-\infty$ to ∞, on the basis of (12) it is possible to construct a family of $\mathbf{p}_2$ trajectories passing through any layer point. The $\mathbf{p}_2$ trajectories for $x_0 = 0, 0.5, 1.0, 1.5$ were constructed in Fig. 4b in a section parallel to the xoy plane. The $\mathbf{p}_2$ trajectories emanate from the origin of coordinates and the tangent to them, representing the $\mathbf{p}_2$ vector, forms the angle

$$\gamma = \text{arctg}(1/\text{sh}(\pi x_0/2))$$

to the y axis.

Much information on the nature of spatial orientation of the $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ vectors is provided by the surface of effect of the main shear stresses and, in particular, the p_{13} stresses. It follows from (8) that the xoy plane and the planes parallel to it are the surfaces of effect of the main shear stresses p_{13}. The direction cosines of the $\mathbf{p}_{13}$ vector, operative over these surfaces, are obtained on the basis of formulas (5):

$$l_{13} = \sqrt{2} \frac{\sin(\pi y/2)\,\text{ch}(\pi x/2)}{\sqrt{\text{ch}\,\pi x - \cos \pi y}},$$
$$m_{13} = -\sqrt{2} \frac{\cos(\pi y/2)\,\text{sh}(\pi x/2)}{\sqrt{\text{ch}\,\pi x - \cos \pi y}}, \quad n_{13} = 0. \quad (14)$$

Expressions (14) make it possible to write an equation for the trajectories of the $\mathbf{p}_{13}$ vector:

$$y_{13} = \frac{2}{\pi} \arccos\left(\cos\left(\frac{\pi}{2} y_0\right) \text{ch}\left(\frac{\pi}{2} x_{13}\right)\right).$$

Here $x_0 = 0$ and y_0 are the coordinates of the point on the xoy plane through which the $\mathbf{p}_{13}$ trajectory passes. As follows from (8) and (14), the families of trajectories $\mathbf{p}_2$ and $\mathbf{p}_{13}$ form a mutually orthogonal system. Figure 4b shows a section of the layer by two pairs of planes parallel to the xoy and yoz planes. Within the cutoff volumes the trajectories of the $\mathbf{p}_2$ vector, passed through straight lines with the coordinates $x = 0, 0.5, 1, 1.5$, $y = 1$, form cylindrical surfaces cutting the bottom of the layer along the z axis. We note that with $x_0 = 0$ the cylindrical surface degenerates into the yoz plane. Since the generatrix of the cylindrical surfaces is parallel to z and divides the angle between the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ in half (8), the plane tangent to these surfaces and normal to the xoy plane will represent a surface along which the greatest of the main shear stresses p_{13} is operative. The $\mathbf{p}_{13}$ vector here is directed parallel to the z axis. In connection with what has been said above, surfaces with $\mathbf{p}_2$ trajectories can also be regarded as surfaces of p_{13} operation. The second pair of surfaces of operation of $\mathbf{p}_{13}$, adjoint to the surfaces of the $\mathbf{p}_2$ trajectories, will be, as was already noted earlier, surfaces parallel to xoy. The trajectories of the main shear stresses $\mathbf{p}_{13}$, operative along these surfaces, passing through points on the axis $y = 0.4, 0.8$, are constructed in Fig. 4b.

So much attention was devoted to the construction and analysis of the nature of behavior of the trajectories of the main stresses because they were determined for the first time with this mechanism and are important for understanding secondary processes accompanying longitudinal shear, such as echelonlike folding and fracture formation. A comparison of the field of stresses obtained from the solution constructed above and the fields of rates of deformations measurable during physical modeling at the surface of plastic models will make it possible to supplement our ideas concerning the nature of the deformation process in regions of active shears.

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